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Journal of Computational Physics 207 (2005) 427–456
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A mixed local discontinuous Galerkin method for a class of nonlinear problems in fluid mechanics ☆

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Received 25 October 2004; received in revised form 18 January 2005; accepted 25 January 2005
Available online 17 March 2005

Abstract

In this paper, we present and analyze a new mixed local discontinuous Galerkin (LDG) method for a class of nonlinear model that appears in quasi-Newtonian Stokes fluids. The approach is based on the introduction of the flux and the tensor gradient of the velocity as further unknowns. In addition, a suitable Lagrange multiplier is needed to ensure that the corresponding discrete variational formulation is well posed. This yields a two-fold saddle point operator equation as the resulting LDG mixed formulation, which is then reduced to a dual mixed formulation. Applying a nonlinear version of the well known Babuška–Brezzi theory, we prove that the discrete formulation is well posed and derive the corresponding a priori error analysis. We also develop a reliable a-posteriori error estimate and propose the associated adaptive algorithm to compute the finite element solutions. Finally, several numerical results illustrate the performance of the method and confirm its capability to localize boundary and inner layers, as well as singularities.

1. Introduction

Nowadays, the discontinuous Galerkin (DG) methods are widely used to solve diverse problems in physics and engineering sciences (see [1] and the references therein for an overview). This is mainly due to the fact that no interelement continuity is required for these methods, which is attractive to be analized in the frame of h, p and $h-p$ versions. Indeed, there are many applications of these approaches to different kind

☆ This research was partially supported by CONICYT-Chile through the FONDAP Program in Applied Mathematics, and by the Dirección de Investigación of the Universidad de Concepción through the Advanced Research Groups Program.
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0021-9991/$ - see front matter
doi:10.1016/j.jcp.2005.01.017

of linear elliptic problems, such as the Stokes, Maxwell and Oseen equations (see, e.g., [10–12,22]). The utilization of DG methods to numerically solve nonlinear boundary value problems has been considered only lately, and to the best of our knowledge, the first results in this direction can be found in [7,18]. More precisely, we developed in [7] the extension of the local discontinuous Galerkin (LDG) method to a class of nonlinear diffusion problems, whereas the extension of the interior penalty *hp* DG method to quasilinear elliptic equations was studied in [18].

On the other hand, in connection with a-posteriori error analysis for discontinuous Galerkin methods, we first refer to [3,23], where residual estimators for the L^2-norm of the error and implicit estimators based on local problems for the energy norm of the error, are provided. In addition, a residual-based reliable a-posteriori error estimate for a mesh dependent energy norm of the error is presented in [2] for a general family of discontinuous Galerkin methods. The procedure from [2], which is valid for any other conservative method, relies on a Helmholtz decomposition of the gradient of the error and applies to nonconvex polyhedra domains in two and three dimensions. More recently, we derived in [6] a new explicit and reliable a posteriori error estimate for the LDG applied to second order elliptic equations in divergence form, including the nonlinear diffusion problems studied in [7]. Similarly as in [2], our analysis there makes use of Helmholtz decompositions, but in contrast to that work, which requires certain polynomial behavior of the Dirichlet datum, we just need to consider a suitable piecewise polynomial function interpolating that boundary condition.

In the present paper, we are interested in the a-priori and a-posteriori error analyses of the LDG method as applied to certain type of nonlinear Stokes models, whose kinematic viscosities are nonlinear monotone functions of the gradient of the velocity. In order to define the boundary value problem explicitly, we first let Ω be a bounded open subset of $\mathbb{R}^2$ with Lipschitz continuous (polygonal) boundary Γ. Then, given $\mathbf{f} \in [L^2(\Omega)]^2$ and $\mathbf{g} \in [H^{1/2}(\Gamma)]^2$, we look for the velocity $\mathbf{u} := (u_1,u_2)^{\mathrm{t}}$ and the pressure p of a fluid occupying the region Ω, such that

$$\begin{aligned} &-\mathbf{div}(\psi(|\nabla\mathbf{u}|)\nabla\mathbf{u} - p\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega, \\ &\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\mathbf{div}$ and div are the usual vector and scalar divergence operators, $\nabla\mathbf{u}$ is the tensor gradient of $\mathbf{u}$, $|\cdot|$ is the euclidean norm of $\mathbb{R}^2$, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is the nonlinear kinematic viscosity function of the fluid. We remark that, as a consequence of the incompressibility of the fluid, the Dirichlet datum $\mathbf{g}$ must satisfy the compatibility condition $\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} = 0$, where $\boldsymbol{\nu}$ is the unit outward normal to Γ. Hereafter, given any Hilbert space S, we denote by S^2 and $S^{2\times 2}$ the spaces of vectors and tensors of order 2, respectively, with entries in S, provided with the product norms induced by the norm of S. Also, for tensors $\mathbf{r} := (r_{ij}), \mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$, and vectors $\mathbf{v} := (v_1, v_2)^{\mathrm{t}}$, $\mathbf{w} := (w_1, w_2)^{\mathrm{t}} \in \mathbb{R}^2$, we use the standard notation $\mathbf{r} : \mathbf{s} := \sum_{i,j=1}^2 r_{ij}s_{ij}$, and denote by $\mathbf{v}\otimes\mathbf{w}$ the tensor of order 2 whose ijth entry is v_iw_j. Note that the following identity holds: $\mathbf{v}\cdot(\mathbf{r}\mathbf{w}) = \mathbf{r}:(\mathbf{v}\otimes\mathbf{w})$.

We now let $\psi_{ij} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ be the mapping given by $\psi_{ij}(\mathbf{r}) := \psi(|\mathbf{r}|)r_{ij}$ $\forall \mathbf{r} := (r_{ij}) \in \mathbb{R}^{2\times 2}$, $\forall i,j \in \{1,2\}$, and define the tensor $\boldsymbol{\psi} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $\boldsymbol{\psi}(\mathbf{r}) := (\psi_{ij}(\mathbf{r}))$ $\forall \mathbf{r} \in \mathbb{R}^{2\times 2}$. Then, throughout this paper we assume that ψ is of class C^1 and that there exist $C_1, C_2 > 0$ such that for all $\mathbf{r} := (r_{ij})$, $\mathbf{s} := (s_{ij}) \in \mathbb{R}^{2\times 2}$, there hold

$$|\psi_{ij}(\mathbf{r})| \leqslant C_1\|\mathbf{r}\|_{\mathbb{R}^{2\times 2}}, \quad \left|\frac{\partial}{\partial r_{kl}}\psi_{ij}\right| \leqslant C_1 \qquad \forall i,j,k,l \in \{1,2\} \tag{1.2}$$

and

$$\sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial r_{kl}} \psi_{ij}(\mathbf{r}) s_{ij} s_{kl} \geqslant C_2 \|\mathbf{s}\|^2_{\mathbb{R}^{2\times 2}}. \tag{1.3}$$

It is important to recall here that the nonlinear model (1.1) for fluids with large stresses was first studied in [21] by using a dual-mixed variational formulation based on inverting the relation $\tilde{\boldsymbol{\sigma}} = \psi(|\nabla\mathbf{u}|)\nabla\mathbf{u}$ to obtain $\nabla\mathbf{u}$ as an explicit function of $\tilde{\boldsymbol{\sigma}}$. We remark, however, that this procedure cannot be applied in all cases since such explicit inversion formula is not always available. Certainly, one could also deal with (1.1) without requiring the inversion of that relation, by applying the usual primal-mixed variational formulation (see, e.g. [17] for the well known linear case). Nevertheless, in this setting the velocity $\mathbf{u}$ lives in the space $[H^1(\Omega)]^2$, and hence the corresponding finite element subspace needs to be a subset of the continuous functions. In addition, the Dirichlet boundary condition, being essential and non-homogeneuous, will necessarily lead to a non-conforming Galerkin scheme.

On the other hand, a dual-mixed formulation of (1.1) not requiring any inversion procedure, and based on low-order finite element subspaces (Raviart–Thomas spaces of order zero to approximate the flux, and piecewise constants to approximate the other unknowns), is proposed in [13,14]. The variables $\mathbf{t} := \nabla\mathbf{u}$ and $\boldsymbol{\sigma} := \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I}$, as well as a Lagrange multiplier ξ, are introduced there as auxiliary unknowns, which yields the continuous formulation: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, p, \mathbf{u}, \xi) \in [L^2(\Omega)]^{2\times 2} \times H(\mathrm{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbb{R}$ *such that*

$$\begin{aligned} &\int_\Omega \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} - \int_\Omega \boldsymbol{\sigma} : \mathbf{s} - \int_\Omega p\, \mathrm{tr}(\mathbf{s}) = 0,\\ &-\int_\Omega \boldsymbol{\tau} : \mathbf{t} - \int_\Omega q\, \mathrm{tr}(\mathbf{t}) - \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) + \xi \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = -\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma,\\ &-\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) + \eta \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \end{aligned} \tag{1.4}$$

for all $(\mathbf{s}, \boldsymbol{\tau}, q, \mathbf{v}, \eta) \in [L^2(\Omega)]^{2\times 2} \times H(\mathrm{div};\Omega) \times L^2(\Omega) \times [L^2(\Omega)]^2 \times \mathbb{R}$.

At this point, we observe that the usual Stokes model is obtained when $\boldsymbol{\psi}(\mathbf{r}) = \psi_0\mathbf{r}\ \forall \mathbf{r} \in \mathbb{R}^{2\times 2}$, where ψ_0 is the constant viscosity of a fluid. The application of the LDG method to this linear problem in the classical velocity–pressure formulation, including the derivation of a priori error estimates for $h - p$ approximations, has been studied in [12,25]. The main advantages of the LDG approach, as compared to the primal-mixed and dual-mixed finite element schemes, are the high order of approximation provided, the high degree of parallelism involved, and, as already mentioned, the suitability for h, p, and $h - p$ refinements (because of the use of arbitrary polynomial degrees on different finite elements). The main disadvantage, however, is the consequent increase of the number of unknowns of the corresponding discrete systems.

In this work, we extend the analysis developed in [7,25], and apply the mixed LDG approach to solve (1.1). We consider regular and conforming meshes made up of straight triangles, and avoid the zero mean value condition on the pressure by means of a suitable Lagrange multiplier. The rest of the paper is organized as follows. In Section 2 we introduce the full mixed local discontinuous Galerkin scheme, which includes the definition of the corresponding numerical fluxes and the reduced mixed formulation. In Section 3 we show the unique solvability of the mixed LDG scheme and derive the Céa-type error estimates. In contrast to the analysis presented in [13], we only need piecewise discontinuous polynomials to approximate the unknowns. The usual a-priori error estimates in energy and L^2 norms are proved in Section 4. Next, in Section 5 we follow the approach given in [19] and deduce an a-posteriori estimate for the error measured in the energy norm. Finally, some numerical experiments validating the good performance of the associated adaptive algorithm are reported in Section 6. We even consider here meshes with hanging nodes, whose analysis is not covered yet by the present theory.

2. The mixed LDG formulation

We follow [13] and introduce the tensor gradient $\mathbf{t} := \nabla\mathbf{u}$ in Ω, and the flux $\boldsymbol{\sigma} := \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I}$ in Ω as additional unknowns. Since $\operatorname{div}\mathbf{u} = \operatorname{tr}(\nabla\mathbf{u})$, the incompressibility condition can be rewritten as $\operatorname{tr}(\mathbf{t}) = 0$ in Ω. In this way, (1.1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p)$ *in appropriate spaces such that, in the distributional sense,*

$$\begin{aligned} &\mathbf{t} = \nabla\mathbf{u} \quad \text{in } \Omega, \qquad \boldsymbol{\sigma} = \boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I} \quad \text{in } \Omega, \qquad -\mathbf{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega, \\ &\operatorname{tr}(\mathbf{t}) = 0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma. \end{aligned} \tag{2.1}$$

As in [7], we now let $\mathcal{T}_h$ be a shape-regular triangulation of $\bar{\Omega}$ made up of straight triangles T with diameter h_T and unit outward normal to ∂T given by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max_{T\in\mathcal{T}_h} h_T$. In addition, we define the edges of $\mathcal{T}_h$ as follows. An *interior edge of* $\mathcal{T}_h$ *is the* (non-empty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial T \cap \Gamma$, where T is a boundary element of $\mathcal{T}_h$. We denote by $\mathcal{E}_I$ and $\mathcal{E}_D$ the union of all interior and boundary edges, respectively, of $\mathcal{T}_h$, and set $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_D$ the union of all edges of $\mathcal{T}_h$. Further, for each edge $e \subseteq \mathcal{E}$, h_e represents its length. Also, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, that is there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \leqslant \frac{h_T}{h_{T'}} \leqslant l$ for each pair $T, T' \in \mathcal{T}_h$ sharing an interior edge.

The LDG variational formulation is described next. We first multiply the first fourth equations of (2.1) by smooth test functions $\boldsymbol{\tau}$, $\mathbf{s}$, $\mathbf{v}$ and q, respectively, integrate by parts over each $T \in \mathcal{T}_h$, and obtain

$$\begin{aligned} &\int_T \boldsymbol{\psi}(\mathbf{t}) : \mathbf{s} - \int_T \boldsymbol{\sigma} : \mathbf{s} - \int_T p\,\operatorname{tr}(\mathbf{s}) = 0, \\ &\int_T \mathbf{t} : \boldsymbol{\tau} + \int_T \mathbf{u}\cdot\operatorname{div}\boldsymbol{\tau} - \int_{\partial T} \boldsymbol{\tau} : \mathbf{u}\otimes\boldsymbol{\nu}_T = 0, \\ &\int_T q\,\operatorname{tr}(\mathbf{t}) = 0, \\ &\int_T \boldsymbol{\sigma} : \nabla\mathbf{v} - \int_{\partial T} \boldsymbol{\sigma} : \mathbf{v}\otimes\boldsymbol{\nu}_T = \int_T \mathbf{f}\cdot\mathbf{v}. \end{aligned} \tag{2.2}$$

Then, given $k \in \mathbf{N}$ and $r = k$ or $r = k - 1$, we want to approximate the exact solution $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p)$ by discrete functions $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, p_h)$ in the finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times W_h$, where

$$\begin{aligned} \boldsymbol{\Sigma}_h &:= \left\{ \mathbf{t}_h \in [L^2(\Omega)]^{2\times 2} : \ \mathbf{t}_h\big|_T \in [\mathbb{P}_r(T)]^{2\times 2} \ \ \forall T \in \mathcal{T}_h \right\}, \\ \mathbf{V}_h &:= \left\{ \mathbf{v}_h \in [L^2(\Omega)]^2 : \ \mathbf{v}_h\big|_T \in [\mathbb{P}_k(T)]^2 \ \ \forall T \in \mathcal{T}_h \right\}, \\ W_h &:= \left\{ q_h \in L^2(\Omega) : \ q_h\big|_T \in \mathbb{P}_{k-1}(T) \ \ \forall T \in \mathcal{T}_h \right\}. \end{aligned} \tag{2.3}$$

Hereafter, given an integer $m \geqslant 0$ we denote by $\mathbb{P}_m(T)$ the space of polynomials of total degree at most m on T. Also, the spaces $\boldsymbol{\Sigma}_h$ and W_h are endowed with the usual product norms of $[L^2(\Omega)]^{2\times 2}$ and $L^2(\Omega)$, which are denoted by $\|\cdot\|_{[L^2(\Omega)]^{2\times 2}}$ and $\|\cdot\|_{L^2(\Omega)}$, respectively. The norm for $\mathbf{V}_h$ will be defined later on in Section 3.

We recall that the idea of the LDG method is to enforce the conservation laws given in (2.2) with the traces of $\boldsymbol{\sigma}$ and $\mathbf{u}$ on the boundary of each $T \in \mathcal{T}_h$ being replaced by suitable numerical approximations of them. In other words, we consider the following formulation: *Find* $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, p_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times W_h$ *such that on each* $T \in \mathcal{T}_h$ *there hold*

$$
\begin{aligned}
&\int_T \psi(\mathbf{t}_h) : \mathbf{s}_h - \int_T \boldsymbol{\sigma}_h : \mathbf{s}_h - \int_T p_h \operatorname{tr}(\mathbf{s}_h) = 0,\\
&\int_T \mathbf{t}_h : \boldsymbol{\tau}_h + \int_T \mathbf{u}_h \cdot \operatorname{div} \boldsymbol{\tau}_h - \int_{\partial T} \boldsymbol{\tau}_h : \widehat{\mathbf{u}} \otimes \boldsymbol{\nu}_T = 0,\\
&\int_T q_h \operatorname{tr}(\mathbf{t}_h) = 0,\\
&\int_T \boldsymbol{\sigma}_h : \nabla \mathbf{v}_h - \int_{\partial T} \widehat{\boldsymbol{\sigma}} : \mathbf{v}_h \otimes \boldsymbol{\nu}_T = \int_T \mathbf{f} \cdot \mathbf{v}_h
\end{aligned}
\tag{2.4}
$$

for all $(\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, q_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times W_h$, where the numerical fluxes $\widehat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on $\mathbf{u}_h$, $\boldsymbol{\sigma}_h$, and the boundary conditions, are chosen so that some compatibility conditions are satisfied.

Then, we define the *average* and the *jump* of $q := (q_T)_{T\in\mathscr{T}_h} \in \prod_{T\in\mathscr{T}_h} L^2(T)$ across $e \subseteq \mathscr{E}_I$ by

$$\{q\} := \tfrac{1}{2}\big(q_{T,e} + q_{T',e}\big) \quad \text{and} \quad [\![q]\!] := q_{T,e}\boldsymbol{\nu}_T + q_{T',e}\boldsymbol{\nu}_{T'}, \tag{2.5}$$

where $q_{T,e}$ and $q_{T',e}$ denote, respectively, the restrictions of q_T and $q_{T'}$ to e. Analogously, the corresponding *average* and *jump* of $\boldsymbol{\zeta} := (\boldsymbol{\zeta}_T)_{T\in\mathscr{T}_h} \in \prod_{T\in\mathscr{T}_h} [L^2(T)]^{2\times 2}$ are defined by

$$\{\boldsymbol{\zeta}\} := \tfrac{1}{2}\big(\boldsymbol{\zeta}_{T,e} + \boldsymbol{\zeta}_{T',e}\big) \quad \text{and} \quad [\![\boldsymbol{\zeta}]\!] := \boldsymbol{\zeta}_{T,e}\boldsymbol{\nu}_T + \boldsymbol{\zeta}_{T',e}\boldsymbol{\nu}_{T'}. \tag{2.6}$$

Finally, for any $\mathbf{v} := (\mathbf{v}_T)_{T\in\mathscr{T}_h} \in \prod_{T\in\mathscr{T}_h} [L^2(T)]^2$, we let its average and jump across $e \subseteq \mathscr{E}_I$ by

$$\{\mathbf{v}\} := \tfrac{1}{2}\big(\mathbf{v}_{T,e} + \mathbf{v}_{T',e}\big) \quad \text{and} \quad [\![\mathbf{v}]\!] := \mathbf{v}_{T,e} \cdot \boldsymbol{\nu}_T + \mathbf{v}_{T',e} \cdot \boldsymbol{\nu}_{T'}, \tag{2.7}$$

and introduce its *tensorial* jump by

$$\underline{[\![\mathbf{v}]\!]} := \mathbf{v}_{T,e} \otimes \boldsymbol{\nu}_T + \mathbf{v}_{T',e} \otimes \boldsymbol{\nu}_{T'}. \tag{2.8}$$

We notice that for any $e \subseteq \mathscr{E}_D$, the traces on e of every scalar, vector and tensor functions $q \in \prod_{T\in\mathscr{T}_h} L^2(T), \mathbf{v} \in \prod_{T\in\mathscr{T}_h} [L^2(T)]^2$, and $\boldsymbol{\zeta} \in \prod_{T\in\mathscr{T}_h} [L^2(T)]^{2\times 2}$, respectively, are uniquely defined, and hence we set

$$\{q\} := q, \qquad \{\mathbf{v}\} := \mathbf{v} \quad \text{and} \quad \{\boldsymbol{\zeta}\} := \boldsymbol{\zeta},$$

as well as

$$[\![q]\!] := q\boldsymbol{\nu}_T, \qquad [\![\mathbf{v}]\!] := \mathbf{v} \cdot \boldsymbol{\nu}_T, \qquad \underline{[\![\mathbf{v}]\!]} := \mathbf{v} \otimes \boldsymbol{\nu}_T \quad \text{and} \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\, \boldsymbol{\nu}_T.$$

We are now ready to complete the mixed LDG formulation (2.4). Indeed, using the approach from [10,12,25] (see also [7]), we define the numerical fluxes $\widehat{\mathbf{u}}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathscr{T}_h$, as follows:

$$\widehat{\mathbf{u}}_{T,e} := \begin{cases} \{\mathbf{u}_h\} + \underline{[\![\mathbf{u}_h]\!]}\boldsymbol{\beta} & \text{if } e \subseteq \mathscr{E}_I, \\ \mathbf{g} & \text{if } e \subseteq \mathscr{E}_D \end{cases} \tag{2.9}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \alpha\underline{[\![\mathbf{u}_h]\!]} & \text{if } e \subseteq \mathscr{E}_I, \\ \boldsymbol{\sigma}_h - \alpha(\mathbf{u}_h - \mathbf{g}) \otimes \boldsymbol{\nu} & \text{if } e \subseteq \mathscr{E}_D, \end{cases} \tag{2.10}$$

where the auxiliary functions α (scalar) and $\boldsymbol{\beta}$ (vector), to be chosen appropriately, are single-valued on each edge $e \subseteq \mathscr{E}$. As in [7], these numerical fluxes are *consistent* and *conservative*.

Now, summing up in (2.4) over all the elements $T \in \mathscr{T}_h$, integrating by parts appropriately, using the definitions of the numerical fluxes, and applying some algebraic identities, we arrive to the formulation: *Find* $(\mathbf{t}_h, \mathbf{u}_h, \boldsymbol{\sigma}_h, p_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h$ *such that*

$$\int_\Omega \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h - \int_\Omega \boldsymbol{\sigma}_h : \mathbf{s}_h - \int_\Omega p_h \operatorname{tr}(\mathbf{s}_h) = 0, \tag{2.11}$$

$$\int_\Omega \mathbf{t}_h : \boldsymbol{\tau}_h - \int_\Omega \nabla_h \mathbf{u}_h : \boldsymbol{\tau}_h + \int_{\mathscr{E}_I} (\{\boldsymbol{\tau}_h\} - [\![\boldsymbol{\tau}_h]\!] \otimes \boldsymbol{\beta}) : \underline{[\![\mathbf{u}_h]\!]} + \int_{\mathscr{E}_D} \mathbf{u}_h \cdot \boldsymbol{\tau}_h \boldsymbol{\nu} = \int_{\mathscr{E}_D} \mathbf{g} \cdot \boldsymbol{\tau}_h \boldsymbol{\nu}, \tag{2.12}$$

$$\begin{aligned} &\int_\Omega \boldsymbol{\sigma}_h : \nabla_h \mathbf{v}_h - \int_{\mathscr{E}_I} \underline{[\![\mathbf{v}_h]\!]} : (\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta}) - \int_{\mathscr{E}_D} \mathbf{v}_h \cdot \boldsymbol{\sigma}_h \boldsymbol{\nu} + \int_{\mathscr{E}_I} \alpha \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]} \\ &\quad + \int_{\mathscr{E}_D} \alpha (\mathbf{u}_h \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h + \int_{\mathscr{E}_D} \alpha (\mathbf{g} \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu}) \end{aligned} \tag{2.13}$$

and

$$\int_\Omega q_h \operatorname{tr}(\mathbf{t}_h) = 0 \tag{2.14}$$

for all $(\mathbf{s}_h, \mathbf{v}_h, \boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h$, where ∇_h denotes the piecewise gradient operator.

We notice, however, that the discrete scheme (2.11)–(2.14) is not uniquely solvable since adding $(\mathbf{0}, \mathbf{0} - c\mathbf{I}, c)$ to $(\mathbf{t}_h, \mathbf{u}_h, \boldsymbol{\sigma}_h, p_h)$, for any $c \in \mathbb{R}$, yields further solutions of this problem. Therefore, in order to guarantee uniqueness, we proceed as in [13] and require additionally that $\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) = 0$, which leads the introduction of the Lagrange multiplier $\xi_h \in \mathbb{R}$ as a further unknown. In this way, our formulation becomes the dual–dual system: *Find* $((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\sigma}_h, p_h), \xi_h) \in (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \times (\boldsymbol{\Sigma}_h \times W_h) \times \mathbb{R}$ *such that*

$$\begin{aligned} &A((\mathbf{t}_h, \mathbf{u}_h), (\mathbf{s}_h, \mathbf{v}_h)) + B((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\sigma}_h, p_h)) = F(\mathbf{s}_h, \mathbf{v}_h), \\ &B((\mathbf{t}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, q_h)) + C((\boldsymbol{\tau}_h, q_h), \xi_h) = G(\boldsymbol{\tau}_h, q_h), \\ &C((\boldsymbol{\sigma}_h, p_h), \lambda_h) = 0 \end{aligned} \tag{2.15}$$

for all $((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\tau}_h, q_h), \lambda_h) \in (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \times (\boldsymbol{\Sigma}_h \times W_h) \times \mathbb{R}$, where the semilinear form $A : (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \times (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \to \mathbb{R}$, the bilinear forms $B : (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \times (\boldsymbol{\Sigma}_h \times W_h) \to \mathbb{R}$ and $C : (\boldsymbol{\Sigma}_h \times W_h) \times \mathbb{R} \to \mathbb{R}$, and the functionals $F : (\boldsymbol{\Sigma}_h \times \mathbf{V}_h) \to \mathbb{R}$ and $G : (\boldsymbol{\Sigma}_h \times W_h) \to \mathbb{R}$, are defined by

$$\begin{aligned} A((\mathbf{t}_h, \mathbf{u}_h), (\mathbf{s}_h, \mathbf{v}_h)) &:= \int_\Omega \boldsymbol{\psi}(\mathbf{t}_h) : \mathbf{s}_h + \int_{\mathscr{E}_I} \alpha \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}_h]\!]} + \int_{\mathscr{E}_D} \alpha (\mathbf{u}_h \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu}), \\ B((\mathbf{s}_h, \mathbf{v}_h), (\boldsymbol{\tau}_h, q_h)) &:= -\int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h + \int_\Omega \nabla_h \mathbf{v}_h : \boldsymbol{\tau}_h - \int_\Omega q_h \operatorname{tr}(\mathbf{s}_h) \\ &\quad - \int_{\mathscr{E}_I} \underline{[\![\mathbf{v}_h]\!]} : (\{\boldsymbol{\tau}_h\} - [\![\boldsymbol{\tau}_h]\!] \otimes \boldsymbol{\beta}) - \int_{\mathscr{E}_D} \mathbf{v}_h \cdot \boldsymbol{\tau}_h \boldsymbol{\nu}, \\ C((\boldsymbol{\tau}_h, q_h), \lambda_h) &:= \lambda_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h), \\ F(\mathbf{s}_h, \mathbf{v}_h) &:= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h + \int_{\mathscr{E}_D} \alpha (\mathbf{g} \otimes \boldsymbol{\nu}) : (\mathbf{v}_h \otimes \boldsymbol{\nu}) \end{aligned}$$

and

$$G(\boldsymbol{\tau}_h, q_h) := -\int_{\mathscr{E}_D} \mathbf{g} \cdot \boldsymbol{\tau}_h \boldsymbol{\nu}$$

for all $(\mathbf{t}_h, \mathbf{u}_h, \boldsymbol{\sigma}_h, p_h), (\mathbf{s}_h, \mathbf{v}_h, \boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h$.

We point out that one knows in advance that $\xi_h = 0$. In fact, this follows from the second equation of (2.15) taking $\boldsymbol{\tau}_h = \mathbf{I}$ and $q_h = -1$, and using the compatibility condition satisfied by the Dirichlet datum $\mathbf{g}$. Similarly, taking $\boldsymbol{\tau} = \mathbf{I}$ and $q = -1$ in the continuous formulation (1.4), one also deduces that $\xi = 0$ [13,14].

However, we do keep these artificial unknowns in both formulations since they are needed to insure the symmetry of them.

The unique solvability of (2.15) will be established next by applying a slight generalization of the classical Babuška–Brezzi theory to an equivalent mixed formulation (see (2.26) below) that arises after expressing the unknowns $\boldsymbol{\sigma}_h$ and $\mathbf{t}_h$ in terms of $\mathbf{u}_h$ and the Lagrange multiplier ξ_h. In addition, the derivation of the a-priori error estimates for the unknowns of (2.15) will also be based on the analysis of (2.26). We emphasize, however, that the introduction of this equivalent formulation is just for theoretical purposes and by no means for the explicit computation of the solution of (2.15), which is solved directly.

We first introduce the semi-norm and norm associated to $\mathbf{V}_h$. In fact, as in [8] we let $\mathtt{h} \in L^\infty(\mathscr{E})$ be the function related to the local meshsizes, that is

$$\mathtt{h}(x) := \begin{cases} \min\{h_T, h_{T'}\} & \text{if } x \in \operatorname{int}(\partial T \cap \partial T'), \\ h_T & \text{if } x \in \operatorname{int}(\partial T \cap \Gamma). \end{cases} \tag{2.16}$$

Also, we define $\mathtt{\alpha} \in L^\infty(\mathscr{E})$ as

$$\mathtt{\alpha} := \frac{\widehat{\mathtt{\alpha}}}{\mathtt{h}}, \tag{2.17}$$

and consider $\boldsymbol{\beta} \in [L^\infty(\mathscr{E}_I)]^2$ such that

$$\|\boldsymbol{\beta}\|_{[L^\infty(\mathscr{E}_I)]^2} \leqslant \widehat{\beta}, \tag{2.18}$$

where $\widehat{\mathtt{\alpha}} > 0$ and $\widehat{\beta}$ are independent of the meshsize. Then, we set $\mathbf{V}(h) := \mathbf{V}_h + [H^1(\Omega)]^2$ and define the semi-norm $|\cdot| : \mathbf{V}(h) \to \mathbb{R}$ and the energy norm $|||\cdot|||_h : \mathbf{V}(h) \to \mathbb{R}$, respectively, by

$$|\mathbf{v}|_h^2 := \|\mathtt{\alpha}^{1/2}\underline{[\![\mathbf{v}]\!]}\|^2_{[L^2(\mathscr{E}_I)]^{2\times 2}} + \|\mathtt{\alpha}^{1/2}(\mathbf{v}\otimes\boldsymbol{\nu})\|^2_{[L^2(\mathscr{E}_D)]^{2\times 2}} \quad \forall \mathbf{v} \in \mathbf{V}(h) \tag{2.19}$$

and

$$|||\mathbf{v}|||_h^2 := \|\nabla_h \mathbf{v}\|^2_{[L^2(\Omega)]^{2\times 2}} + |\mathbf{v}|_h^2 \quad \forall \mathbf{v} \in \mathbf{V}(h). \tag{2.20}$$

In addition, we let $S : \mathbf{V}(h) \times \boldsymbol{\Sigma}_h \to \mathbb{R}$ be the bilinear form

$$S(\mathbf{v}, \boldsymbol{\tau}_h) := \int_{\mathscr{E}_I} (\{\boldsymbol{\tau}_h\} - [\![\boldsymbol{\tau}_h]\!] \otimes \boldsymbol{\beta}) : \underline{[\![\mathbf{v}]\!]} + \int_{\mathscr{E}_D} \mathbf{v}\cdot\boldsymbol{\tau}_h\boldsymbol{\nu} \quad \forall (\mathbf{v}, \boldsymbol{\tau}_h) \in \mathbf{V}(h) \times \boldsymbol{\Sigma}_h,$$

and let $\mathbf{G} : \boldsymbol{\Sigma}_h \to \mathbb{R}$ be the linear functional defined by $\mathbf{G}(\boldsymbol{\tau}_h) := \int_{\mathscr{E}_D} \mathbf{g}\cdot\boldsymbol{\tau}_h\boldsymbol{\nu}\ \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h$.

It is easy to show, similarly as for Lemmas 3.3 and 3.4 in [7], that S and $\mathbf{G}$ are bounded. In particular, there exists $C_{\mathbf{S}} > 0$, independent of the meshsize, such that

$$|S(\mathbf{v}, \boldsymbol{\tau}_h)| \leqslant C_{\mathbf{S}} |\mathbf{v}|_h \|\boldsymbol{\tau}_h\|_{[L^2(\Omega)]^{2\times 2}} \quad \forall (\mathbf{v}, \boldsymbol{\tau}_h) \in \mathbf{V}(h) \times \boldsymbol{\Sigma}_h. \tag{2.21}$$

Thus, we let $\mathbf{S} : \mathbf{V}(h) \to \boldsymbol{\Sigma}_h$ be the linear and bounded operator induced by the bilinear form S, that is, given $\mathbf{v} \in \mathbf{V}(h)$, $\mathbf{S}(\mathbf{v})$ is the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathbf{S}(\mathbf{v}) : \boldsymbol{\tau}_h = S(\mathbf{v}, \boldsymbol{\tau}_h) \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h, \tag{2.22}$$

which, according to (2.21), satisfies

$$\|\mathbf{S}(\mathbf{v})\|_{[L^2(\Omega)]^{2\times 2}} \leqslant C_{\mathbf{S}} |\mathbf{v}|_h \quad \forall \mathbf{v} \in \mathbf{V}(h). \tag{2.23}$$

Similarly, in virtue of the Riesz Theorem, we let $\mathscr{G}$ be the unique element in $\mathbf{\Sigma}_h$ such that $\int_\Omega \mathscr{G} : \boldsymbol{\tau}_h = \mathbf{G}(\boldsymbol{\tau}_h)\ \forall \boldsymbol{\tau}_h \in \mathbf{\Sigma}_h$. As in [7], we point out that if the exact solution $\mathbf{u}$ of (1.1) is sufficiently smooth, say $\mathbf{u} \in [H^{1+\delta}(\Omega)]^2$, with $\delta > 1/2$, then $\mathbf{S}(\mathbf{u}) = \mathscr{G}$. Actually, this regularity of $\mathbf{u}$ is assumed throughout the rest of the paper.

Now, since $r \geqslant k-1$ and $\nabla_h \mathbf{u}_h|_T \in [\mathbb{P}_{k-1}(T)]^{2\times 2}$ for each $T \in \mathscr{T}_h$, we obtain from the second equation of (2.15) that

$$\mathbf{t}_h = \Pi_{\mathbf{\Sigma}_h}(\nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathscr{G} + \xi_h \mathbf{I}) = \nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathscr{G}, \tag{2.24}$$

whereas (2.11) yields

$$\boldsymbol{\sigma}_h = \Pi_{\mathbf{\Sigma}_h}(\boldsymbol{\psi}(\mathbf{t}_h) - p_h \mathbf{I}) = \Pi_{\mathbf{\Sigma}_h}(\boldsymbol{\psi}(\nabla_h \mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathscr{G}) - p_h \mathbf{I}), \tag{2.25}$$

where $\Pi_{\mathbf{\Sigma}_h}$ stands for the $[L^2(\Omega)]^{2\times 2}$-projection onto $\mathbf{\Sigma}_h$. In this way, employing (2.24) and (2.25), we find that problem (2.15) can be reformulated as: *Find* $((\mathbf{u}_h, \xi_h), p_h) \in (\mathbf{V}_h \times \mathbb{R}) \times W_h$ *such that*

$$\begin{aligned} &[A_h(\mathbf{u}_h, \xi_h), (\mathbf{v}_h, \lambda_h)] + [B_h(\mathbf{v}_h, \lambda_h), p_h] = [F_h, (\mathbf{v}_h, \lambda_h)] \quad \forall (\mathbf{v}_h, \lambda_h) \in \mathbf{V}_h \times \mathbb{R}, \\ &[B_h(\mathbf{u}_h, \xi_h), q_h] = [G_h, q_h] \quad \forall q_h \in W_h, \end{aligned} \tag{2.26}$$

where the operators $A_h : (\mathbf{V}(h) \times \mathbb{R}) \to (\mathbf{V}(h) \times \mathbb{R})'$ and $B_h : (\mathbf{V}(h) \times \mathbb{R}) \to W'$, with $W = L^2(\Omega)$, and the functionals $F_h : \mathbf{V}(h) \times \mathbb{R} \to \mathbb{R}$ and $G_h : W \to \mathbb{R}$, are defined by

$$\begin{aligned} [A_h(\mathbf{w}, \eta), (\mathbf{v}, \lambda)] : = &\int_\Omega \boldsymbol{\psi}(\nabla_h \mathbf{w} - \mathbf{S}(\mathbf{w}) + \mathscr{G} + \eta \mathbf{I}) : (\nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v}) + \lambda \mathbf{I})] \\ &+ \int_{\mathscr{E}_I} \alpha \underline{[\![\mathbf{w}]\!]} : \underline{[\![\mathbf{v}]\!]} + \int_{\mathscr{E}_D} \alpha (\mathbf{w} \otimes \boldsymbol{v}) : (\mathbf{v} \otimes \boldsymbol{v}), \end{aligned} \tag{2.27}$$

$$[B_h(\mathbf{v}, \lambda), q] := -\int_\Omega q \operatorname{div}_h \mathbf{v} + \int_\Omega (q\mathbf{I}) : (\mathbf{S}(\mathbf{v}) - \lambda \mathbf{I}), \tag{2.28}$$

$$[F_h, (\mathbf{v}, \lambda)] := \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_{\mathscr{E}_D} \alpha (\mathbf{g} \otimes \boldsymbol{v}) : (\mathbf{v} \otimes \boldsymbol{v}),$$

$$[G_h, q] := \int_{\mathscr{E}_D} q\, \mathbf{g} \cdot \boldsymbol{v}$$

for all $(\mathbf{w}, \eta), (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}$ and for all $q \in W$. Hereafter, div_h denotes the piecewise divergence operator and $[\cdot,\cdot]$ stands for the corresponding duality pairings.

We remark that B_h, F_h, and G_h, are all bounded with respect to the corresponding norms. In particular, the boundedness of B_h makes use of (2.23), and the boundedness of the functionals F_h and G_h is established in the following lemma.

Lemma 2.1. *There exist* $C_F, C_G > 0$, *depending on* $\widehat{\alpha}, l$ *and* k, *but independent of the meshsize*, *such that*

$$|[F_h, (\mathbf{v}_h, \lambda_h)]| \leqslant C_F \mathscr{B}(\mathbf{f}, \mathbf{g}) \|(\mathbf{v}_h, \lambda_h)\|_{\mathbf{V}(h)\times\mathbb{R}} \quad \forall (\mathbf{v}_h, \lambda_h) \in \mathbf{V}_h \times \mathbb{R} \tag{2.29}$$

and

$$|[G_h, q_h]| \leqslant C_G \|\alpha^{1/2} \mathbf{g} \cdot \boldsymbol{v}\|_{L^2(\mathscr{E}_D)} \|q_h\|_{L^2(\Omega)} \quad \forall q_h \in W_h, \tag{2.30}$$

where

$$\mathscr{B}(\mathbf{f}, \mathbf{g}) := \left\{ \|\mathbf{f}\|^2_{[L^2(\Omega)]^2} + \|\alpha^{1/2} \mathbf{g} \otimes \boldsymbol{v}\|^2_{[L^2(\mathscr{E}_D)]^{2\times 2}} \right\}^{1/2}.$$

Proof. It is similar to the proof of Lemma 4.4 in [7]. □

3. Solvability of the mixed LDG formulation

In this section, we establish the unique solvability of (2.26) and the associated Céa-type error estimate. Besides the already established boundedness of B_h, F_h, and G_h, our analysis requires also to show that A_h is Lipschitz-continuous and strongly monotone, and that B_h satisfies the discrete inf-sup condition. To this end, we first let $X := [L^2(\Omega)]^{2\times 2}$ and introduce the pure nonlinear operator $\mathcal{N} : X \to X'$ forming part of (1.4), that is

$$[\mathcal{N}(\mathbf{r}), (\mathbf{s})] := \int_\Omega \boldsymbol{\psi}(\mathbf{r}) : \mathbf{s} \quad \forall \mathbf{r}, \mathbf{s} \in X. \tag{3.1}$$

We observe that $\mathcal{N}$ is Gâteaux differentiable at each $\tilde{\mathbf{r}} \in X$. In fact, this derivative can be seen as the bounded bilinear form $D\mathcal{N}(\tilde{\mathbf{r}}) : X \times X \to \mathbb{R}$ given by

$$D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) := \int_\Omega D\boldsymbol{\psi}(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s}) = \int_\Omega \left\{ \sum_{i,j,k,l=1}^{2} \frac{\partial}{\partial \tilde{r}_{kl}} \psi_{ij}(\tilde{\mathbf{r}}) r_{kl} s_{ij} \right\} \quad \forall \mathbf{r}, \mathbf{s} \in X, \tag{3.2}$$

where $D\boldsymbol{\psi}(\tilde{\mathbf{r}}) : X \times X \to \mathbb{R}$ is the Gâteaux derivative of $\boldsymbol{\psi}$. It follows, according to (1.2) and (1.3), that there exist positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that

$$|D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{r}, \mathbf{s})| \leqslant \tilde{C}_1 \|\mathbf{r}\|_X \|\mathbf{s}\|_X \quad \text{and} \quad D\mathcal{N}(\tilde{\mathbf{r}})(\mathbf{s}, \mathbf{s}) \geqslant \tilde{C}_2 \|\mathbf{s}\|_X^2 \qquad \forall \tilde{\mathbf{r}}, \mathbf{r}, \mathbf{s} \in X, \tag{3.3}$$

which implies the strong monotonicity and Lipschitz continuity of the operator $\mathcal{N}$ on X.

Next, we introduce the application $\boldsymbol{\varphi} : \mathbf{V}(h) \times \mathbb{R} \to X$ given by

$$\boldsymbol{\varphi}(\mathbf{v}, \lambda) := \nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v}) - \lambda \mathbf{I} \quad \forall (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}, \tag{3.4}$$

so that the corresponding non-linear part $\mathcal{N}_h : (\mathbf{V}(h) \times \mathbb{R}) \to (\mathbf{V}(h) \times \mathbb{R})'$ of A_h is defined by

$$[\mathcal{N}_h(\mathbf{w}, \eta), (\mathbf{v}, \lambda)] := [\mathcal{N}(\boldsymbol{\varphi}(\mathbf{w}, \eta) + \mathcal{G}), \boldsymbol{\varphi}(\mathbf{v}, \lambda)] \quad \forall (\mathbf{w}, \eta), (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}. \tag{3.5}$$

We remark that $\mathcal{N}_h$ also admits a Gâteaux derivative at each $(\mathbf{z}, \zeta) \in \mathbf{V}(h) \times \mathbb{R}$, which can be seen as the bounded bilinear form $D\mathcal{N}_h(\mathbf{z}, \zeta) : (\mathbf{V}(h) \times \mathbb{R}) \times (\mathbf{V}(h) \times \mathbb{R}) \to \mathbb{R}$ given by

$$D\mathcal{N}_h(\mathbf{z}, \zeta)((\mathbf{w}, \eta), (\mathbf{v}, \lambda)) := D\mathcal{N}(\boldsymbol{\varphi}(\mathbf{z}, \zeta) + \mathcal{G})(\boldsymbol{\varphi}(\mathbf{w}, \eta), \boldsymbol{\varphi}(\mathbf{v}, \lambda)) \tag{3.6}$$

for all $(\mathbf{w}, \eta), (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}$. Hence, the Gâteaux derivative of A_h at $(\mathbf{z}, \zeta) \in \mathbf{V}(h) \times \mathbb{R}$ reduces to the bounded bilinear form $DA_h(\mathbf{z}, \zeta) : (\mathbf{V}(h) \times \mathbb{R}) \times (\mathbf{V}(h) \times \mathbb{R}) \to \mathbb{R}$ defined by

$$DA_h(\mathbf{z}, \zeta)((\mathbf{w}, \eta), (\mathbf{v}, \lambda)) := D\mathcal{N}(\boldsymbol{\varphi}(\mathbf{z}, \zeta) + \mathcal{G})(\boldsymbol{\varphi}(\mathbf{w}, \eta), \boldsymbol{\varphi}(\mathbf{v}, \lambda)) + \int_{\mathcal{E}_I} \alpha \underline{[\![\mathbf{w}]\!]} : \underline{[\![\mathbf{v}]\!]} + \int_{\mathcal{E}_D} \alpha (\mathbf{w} \otimes \boldsymbol{\nu}) : (\mathbf{v} \otimes \boldsymbol{\nu}) \tag{3.7}$$

for all $(\mathbf{w}, \eta), (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}$.

On the other hand, taking into account that

$$\int_\Omega \mathbf{I} : (\nabla_h \mathbf{w} - \mathbf{S}(\mathbf{w})) = 0 \quad \forall \mathbf{w} \in \mathbf{V}(h), \tag{3.8}$$

we find that

$$\|\boldsymbol{\varphi}(\mathbf{v}, \lambda)\|_X^2 = \|\nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} + 2|\Omega||\lambda|^2 \quad \forall (\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}. \tag{3.9}$$

In this way, (3.3), (3.6), (3.7) and (3.9) allow us to show that the nonlinear operator A_h is indeed Lipschitz continuous and strongly monotone with respect to the norm

$$\|(\mathbf{w},\zeta)\|^2_{\mathbf{V}(h)\times\mathbb{R}} = |||\mathbf{w}|||_h^2 + |\zeta|^2 \quad \forall (\mathbf{w},\zeta)\in \mathbf{V}(h)\times\mathbb{R}.$$

More precisely, we have the following lemma whose proof is very similar to those of Lemmas 4.1 and 4.2 in [7].

Lemma 3.1. *There exist* $C_{\mathrm{LC}} > 0$ *and* $C_{\mathrm{SM}} > 0$, *independent of the meshsize, such that*

$$\|A_h(\mathbf{w},\zeta) - A_h(\mathbf{v},\lambda)\|_{(\mathbf{V}(h)\times\mathbb{R})'} \leqslant C_{\mathrm{LC}}\|(\mathbf{w}-\mathbf{v},\zeta-\lambda)\|_{\mathbf{V}(h)\times\mathbb{R}}$$

and

$$[A_h(\mathbf{w},\zeta) - A_h(\mathbf{v},\lambda),(\mathbf{w}-\mathbf{v},\zeta-\lambda)] \geqslant C_{\mathrm{SM}}\|(\mathbf{w}-\mathbf{v},\zeta-\lambda)\|^2_{\mathbf{V}(h)\times\mathbb{R}}$$

for all $(\mathbf{w},\zeta),(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbb{R}$.

Our next goal is to show the discrete inf-sup condition of the bilinear form B_h. For this purpose we now let $L^2_0(\Omega)$ be the subspace of functions in $L^2(\Omega)$ with zero mean value, and note that $L^2(\Omega) = L^2_0(\Omega)\oplus\mathbb{R}$, i.e., each $q\in L^2(\Omega)$ can be uniquely decomposed as $q=\tilde{q}+\bar{q}$, with $\tilde{q} := \left(q - \frac{1}{|\Omega|}\int_\Omega q\right)\in L^2_0(\Omega)$ and $\bar{q} := \frac{1}{|\Omega|}\int_\Omega q \in\mathbb{R}$. In addition, it follows easily that

$$\|q\|^2_{L^2(\Omega)} = \|\tilde{q}\|^2_{L^2(\Omega)} + |\Omega|\bar{q}^2. \tag{3.10}$$

We now proceed as in Section 6.5 of [25] and establish the following result.

Lemma 3.2. *There exists a constant* $C_{\mathrm{I}} > 0$, *independent of the meshsize, such that,*

$$\sup_{\mathbf{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}} \frac{[B_h(\mathbf{v}_h,0),r_h]}{|||\mathbf{v}_h|||_h} \geqslant C_{\mathrm{I}}\|r_h\|_{L^2(\Omega)} \quad \forall r_h\in W_h\cap L^2_0(\Omega).$$

Proof. Let $\Pi : [H^1(\Omega)]^2 \to \mathbf{V}_h$ be the Raviart–Thomas equilibrium interpolation operator of degree $k-1$ [4,24]. It is well known that $\Pi\mathbf{w}\in H(\mathrm{div};\Omega)$ $\forall\mathbf{w}\in[H^1(\Omega)]^2$, which implies that its normal components are continuous across the inter-element boundaries, and hence, when $\mathbf{w}\in[H^1_0(\Omega)]^2$ we easily find that $[\![\Pi\mathbf{w}]\!] = 0$ on $\mathcal{E}$. Thus, simple algebraic computations yields

$$[B_h(\Pi\mathbf{w},0),r_h] = -\int_\Omega r_h \,\mathrm{div}\,\mathbf{w}, \quad \forall r_h\in W_h\cap L^2_0(\Omega),$$

and similarly as in Lemma 6.11 from [25] we obtain

$$|||\Pi\mathbf{w}|||_h \leqslant C\|\nabla\mathbf{w}\|_{[L^2(\Omega)]^2},$$

where $C > 0$ is independent of the meshsize. The rest of the proof reduces to apply a continuous inf-sup condition satisfied by B_h together with the Fortin property. □

The discrete inf-sup condition satisfied by the operator B_h is proved next.

Lemma 3.3. *There exists a constant* $C_{\mathrm{INF}} > 0$, *independent of the meshsize, such that,*

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbb{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{\|(\mathbf{v}_h,\lambda_h)\|_{\mathbf{V}(h)\times\mathbb{R}}} \geqslant C_{\mathrm{INF}}\|q_h\|_{L^2(\Omega)} \quad \forall q_h\in W_h.$$

Proof. Let $q_h\in W_h\subseteq L^2(\Omega)$. Since $L^2(\Omega) = L^2_0(\Omega)\oplus\mathbb{R}$, there exists $\tilde{q}_h\in W_h\cap L^2_0(\Omega)$ and $\bar{q}_h\in\mathbb{R}$ such that $q_h = \tilde{q}_h + \bar{q}_h$. Then, applying the linearity of B_h together with (3.8), we have

$$[B_h(\mathbf{v}_h,\lambda_h),q_h] = [B_h(\mathbf{v}_h,\lambda_h),\tilde{q}_h] - 2\bar{q}_h\lambda_h|\Omega| \quad \forall(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbb{R}, \tag{3.11}$$

and hence

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbb{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{\|(\mathbf{v}_h,\lambda_h)\|_{\mathbf{V}(h)\times\mathbb{R}}} \geqslant \sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{[B_h(\mathbf{v}_h,0),q_h]}{|||\mathbf{v}_h|||_h} = \sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{V}_h} \frac{[B_h(\mathbf{v}_h,0),\tilde{q}_h]}{|||\mathbf{v}_h|||_h},$$

which, thanks to Lemma 3.2, implies the existence of a constant $C_{\mathrm{I}} > 0$, independent of the meshsize, such that

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbb{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{\|(\mathbf{v}_h,\lambda_h)\|_{\mathbf{V}(h)\times\mathbb{R}}} \geqslant C_{\mathrm{I}}\|\tilde{q}_h\|_{L^2(\Omega)}. \tag{3.12}$$

On the other hand, we also have that

$$\sup_{(\mathbf{0},0)\neq(\mathbf{v}_h,\lambda_h)\in\mathbf{V}_h\times\mathbb{R}} \frac{[B_h(\mathbf{v}_h,\lambda_h),q_h]}{\|(\mathbf{v}_h,\lambda_h)\|_{\mathbf{V}(h)\times\mathbb{R}}} \geqslant \sup_{0\neq\lambda_h\in\mathbb{R}} \frac{[B_h(\mathbf{0},\lambda_h),q_h]}{|\lambda_h|} \geqslant \frac{[B_h(\mathbf{0},-\bar{q}_h),q_h]}{|\bar{q}_h|} = 2|\Omega||\bar{q}_h|,$$

which, together with (3.12) and (3.10), completes the proof. □

We now let $\|\cdot\|_{\mathrm{LDG}}$ be the norm on $\mathbf{V}(h)\times\mathbb{R}\times L^2(\Omega)$ given by

$$\|(\mathbf{v},\lambda,q)\|^2_{\mathrm{LDG}} := |||\mathbf{v}|||^2_h + |\lambda|^2 + \|q\|^2_{L^2(\Omega)} \quad \forall(\mathbf{v},\lambda,q)\in\mathbf{V}(h)\times\mathbb{R}\times L^2(\Omega).$$

Theorem 3.1. *The LDG scheme* (2.26) *has a unique solution* $(\mathbf{u}_h,\xi_h,p_h)\in\mathbf{V}_h\times\mathbb{R}\times W_h$, *and there exists a constant* $C>0$, *independent of the meshsize, such that*

$$\|(\mathbf{u}_h,\xi_h,p_h)\|_{\mathrm{LDG}} \leqslant C\Big(\mathscr{B}(\mathbf{f},\mathbf{g}) + \|\alpha^{1/2}\mathbf{g}\cdot\mathbf{v}\|_{L^2(\mathscr{E}_D)}\Big). \tag{3.13}$$

Moreover, denoting by C_{B} *the boundedness constant associated to* B_h, *there hold the Strang-type error estimates*

$$\begin{aligned} |||\mathbf{u}-\mathbf{u}_h|||_h \leqslant{}& \left(1+\frac{C_{\mathrm{LC}}}{C_{\mathrm{SM}}}\right)\left(1+\frac{C_{\mathrm{B}}}{C_{\mathrm{INF}}}\right)\inf_{\mathbf{v}_h\in\mathbf{V}_h}|||\mathbf{u}-\mathbf{v}_h|||_h + \frac{C_{\mathrm{B}}}{C_{\mathrm{SM}}}\inf_{q_h\in W_h}\|p-q_h\|_{L^2(\Omega)} \\ &+ C_{\mathrm{SM}}^{-1}\sup_{(\mathbf{0},0)\neq(\mathbf{w},\eta)\in\mathbf{V}_h\times\mathbb{R}} \frac{|[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)]|}{\|(\mathbf{w},\eta)\|_{\mathbf{V}(h)\times\mathbb{R}}} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \|p-p_h\|_{L^2(\Omega)} \leqslant{}& \left(1+\frac{C_{\mathrm{B}}}{C_{\mathrm{INF}}}\right)\inf_{q_h\in W_h}\|p-q_h\|_{L^2(\Omega)} + \frac{C_{\mathrm{LC}}}{C_{\mathrm{INF}}}|||\mathbf{u}-\mathbf{u}_h|||_h \\ &+ C_{\mathrm{INF}}^{-1}\sup_{(\mathbf{0},0)\neq(\mathbf{w},\eta)\in\mathbf{V}_h\times\mathbb{R}} \frac{|[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)] + [B_h(\mathbf{w},\eta),p] - [F_h,(\mathbf{w},\eta)]|}{\|(\mathbf{w},\eta)\|_{\mathbf{V}(h)\times\mathbb{R}}}. \end{aligned} \tag{3.15}$$

Proof. The unique solvability of (2.26) and the upper bound (3.13) follow from Lemmas 3.1, 3.3 and 2.1, and a nonlinear version of the classical Babuška–Brezzi theory (see, e.g., Lemma 2.1 in [15]), whereas the derivation of the Strang-type error estimates is a simple extension to the present nonlinear case of Propositions 4.1 and 4.3 in [25]. The latter means that (3.14) and (3.15) basically follow from the strong monotonicity and Lipschitz-continuity of A_h, the boundedness of B_h, and the discrete inf-sup condition satisfied by B_h. We omit further details and refer the interested reader to chapter 5 in [5]. □

4. A-priori error analysis

In order to derive the a-priori error estimates for the mixed LDG scheme (2.15), we need some preliminary results. We begin with the following lemma establishing local approximation properties of piecewise polynomials. For the original result dealing with integer indexes we refer to [9], whereas a simple proof for the extension to non-integer Sobolev seminorms can be seen in [16].

Lemma 4.1. *Let $\mathscr{T}_h$ be a regular triangulation and let $T \in \mathscr{T}_h$. Given a non-negative integer m, let $\Pi_T^m : L^2(T) \to \mathbb{P}_m(T)$ be the linear and bounded operator given by the $L^2(T)$-orthogonal projection, which satisfies $\Pi_T^m(p) = p$ for all $p \in \mathbb{P}_m(T)$. Then there exists $C_{\mathrm{ort}} > 0$, independent of the meshsize, such that for each s, t satisfying $0 \leqslant s \leqslant m+1$ and $0 \leqslant s < t$, there holds*

$$|(\mathbf{I} - \Pi_T^m)(w)|_{H^s(T)} \leqslant C_{\mathrm{ort}} h_T^{\min\{t,m+1\}-s} \|w\|_{H^t(T)} \quad \forall w \in H^t(T), \tag{4.1}$$

and for each $t > 1/2$ there holds

$$\|(\mathbf{I} - \Pi_T^m)(w)\|_{L^2(\partial T)} \leqslant C_{\mathrm{ort}} h_T^{\min\{t,m+1\}-1/2} \|w\|_{H^t(T)} \quad \forall w \in H^t(T). \tag{4.2}$$

The analogue of Lemma 3.1 in [7], which provides useful estimates concerning averages and jumps on the edges of the triangulation, is also required.

Lemma 4.2. *There exist constants $\bar{C}_1, \bar{C}_2, > 0$, independent of the meshsize, such that for all $\boldsymbol{\zeta} := (\zeta_T)_{T\in\mathscr{T}_h} \in \prod_{T\in\mathscr{T}_h} [L^2(T)]^{2\times 2}$, there hold*

(i) $\|\mathrm{h}^{1/2}\{\boldsymbol{\zeta}\}\|^2_{[L^2(\mathscr{E}_I)]^{2\times 2}} \leqslant \bar{C}_1 \sum_{T\in\mathscr{T}_h} h_T \|\zeta_T\|^2_{[L^2(\partial T)]^{2\times 2}}$,

(ii) $\|\mathrm{h}^{1/2}[\![\boldsymbol{\zeta}]\!]\|^2_{[L^2(\mathscr{E}_I)]^2} \leqslant \bar{C}_2 \sum_{T\in\mathscr{T}_h} h_T \|\zeta_T\|^2_{[L^2(\partial T)]^{2\times 2}}$.

At this point, we observe that the assumed regularity on the exact solution $\mathbf{u}$ guarantees that the jump $\underline{[\![\mathbf{u}]\!]}$ vanishes on any interior edge of $\mathscr{T}_h$. In addition, since $\boldsymbol{\sigma} = \boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I} \in [L^2(\Omega)]^{2\times 2}$ and $-\mathbf{div}(\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) = \mathbf{f}$ in Ω, with $\mathbf{f} \in [L^2(\Omega)]^2$, we conclude that $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$, whence $[\![\boldsymbol{\sigma}]\!] = 0$ on each $e \in \mathscr{E}_I$. Also, we recall that $\xi = 0$, and $\boldsymbol{\sigma}$ satisfies $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0$, which due to the kind of nonlinearity we are dealing with, is equivalent to the fact that $p \in L_0^2(\Omega)$. On the other hand, besides $\Pi_{\boldsymbol{\Sigma}_h}$, the $[L^2(\Omega)]^{2\times 2}$-projection onto $\boldsymbol{\Sigma}_h$, in what follows we need the operators $\Pi_{\mathbf{V}_h}$ and Π_{W_h}, which denote the $[L^2(\Omega)]^2$ and $L^2(\Omega)$ projections onto $\mathbf{V}_h$ and W_h, respectively. According to the definitions of $\boldsymbol{\Sigma}_h$, $\mathbf{V}_h$, and W_h (see (2.3)) we find that, given $\boldsymbol{\tau} := (\tau_{ij}) \in [L^2(\Omega)]^{2\times 2}$, $\mathbf{v} := (v_i) \in [L^2(\Omega)]^2$, and $q \in L^2(\Omega)$, there hold

$$\Pi_{\boldsymbol{\Sigma}_h}(\boldsymbol{\tau})|_T = (\Pi_T^r(\tau_{ij}|_T)), \quad \Pi_{\mathbf{V}_h}(\mathbf{v})|_T = (\Pi_T^k(v_i|_T)) \quad \text{and} \quad \Pi_{W_h}(q)|_T = \Pi_T^{k-1}(q|_T) \tag{4.3}$$

for all $T \in \mathscr{T}_h$, where $r = k$ or $r = k - 1$.

4.1. Energy norm error estimates

We first provide an upper bound for the consistency term appearing in the Strang type error estimates (3.14) and (3.15) (cf. Theorem 3.1).

Lemma 4.3. *Assume that $\sigma|_T := (\psi(\nabla\mathbf{u}) - p\mathbf{I})|_T \in [H^t(T)]^{2\times 2}$ for all $T \in \mathscr{T}_h$, with $t > 1/2$. Then, there exists $C_{\mathrm{con}} > 0$, independent of the meshsize, but depending on $\widehat{\alpha}, \widehat{\beta}$, and l, such that for each $(\mathbf{w},\eta) \in \mathbf{V}(h) \times \mathbb{R}$, $(\mathbf{w},\eta) \neq (0,0)$,*

$$\frac{|[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)]+[B_h(\mathbf{w},\eta),p]-[F_h,(\mathbf{w},\eta)]|}{|||\mathbf{w}|||_h}\leqslant C_{\rm con}\left\{\sum_{T\in\mathcal{T}_h}h_T^{2\min\{t,r+1\}}\|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times2}}\right\}^{1/2}.$$

Proof. Let $(\mathbf{w},\eta)\in\mathbf{V}(h)\times\mathbb{R}$. Since $\xi=0$, $\mathbf{S}(\mathbf{u})=\mathcal{G}$, $[\![\mathbf{u}]\!]=0$ on $\mathcal{E}_I$, $\mathbf{f}=-\mathbf{div}(\psi(\nabla\mathbf{u})-p\mathbf{I})$ in Ω, and $\mathbf{u}=\mathbf{g}$ on Γ, we find that

$$\begin{aligned}[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)]+[B_h(\mathbf{w},\eta),p]-[F_h,(\mathbf{w},\eta)] &= \int_\Omega\boldsymbol{\psi}(\nabla\mathbf{u}):(\nabla_h\mathbf{w}-\mathbf{S}(\mathbf{w})+\eta\mathbf{I})+\int_{\mathcal{E}_D}\alpha(\mathbf{u}\otimes\boldsymbol{v}):(\mathbf{w}\otimes\boldsymbol{v})\\ &\quad-\int_\Omega p\mathbf{I}:(\nabla_h\mathbf{w}-\mathbf{S}(\mathbf{w})+\eta\mathbf{I})\\ &\quad-\int_\Omega\mathbf{f}\cdot\mathbf{w}-\int_{\mathcal{E}_D}\alpha(\mathbf{g}\otimes\boldsymbol{v}):(\mathbf{w}\otimes\boldsymbol{v})\\ &=\int_\Omega\boldsymbol{\psi}(\nabla\mathbf{u}):(\nabla_h\mathbf{w}-\mathbf{S}(\mathbf{w}))+\int_\Omega\mathbf{w}\cdot\mathbf{div}(\boldsymbol{\psi}(\nabla\mathbf{u})-p\mathbf{I})\\ &\quad-\int_\Omega p\mathbf{I}:(\nabla_h\mathbf{w}-\mathbf{S}(\mathbf{w}))+\eta\int_\Omega\mathrm{tr}(\boldsymbol{\sigma})\\ &=\int_\Omega\boldsymbol{\sigma}:(\nabla_h\mathbf{w}-\mathbf{S}(\mathbf{w}))+\int_\Omega\mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\sigma}.\end{aligned}\tag{4.4}$$

Applying Gauss' formula on each element $T\in\mathcal{T}_h$, we obtain

$$\begin{aligned}\int_\Omega\mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\sigma}&=\sum_{T\in\mathcal{T}_h}\int_T\mathbf{w}\cdot\mathbf{div}\,\boldsymbol{\sigma}=\sum_{T\in\mathcal{T}_h}\left(-\int_T\boldsymbol{\sigma}:\nabla\mathbf{w}+\int_{\partial T}\mathbf{w}\cdot\boldsymbol{\sigma}\boldsymbol{v}\right)\\ &=-\int_\Omega\boldsymbol{\sigma}:\nabla_h\mathbf{w}+\int_{\mathcal{E}_I}\{\boldsymbol{\sigma}\}:\underline{[\![\mathbf{w}]\!]}+\int_{\mathcal{E}_D}\mathbf{w}\cdot\boldsymbol{\sigma}\boldsymbol{v},\end{aligned}$$

which, replaced back into (4.4), yields

$$[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)]+[B_h(\mathbf{w},\eta),p]-[F_h,(\mathbf{w},\eta)]=-\int_\Omega\mathbf{S}(\mathbf{w}):\boldsymbol{\sigma}+\int_{\mathcal{E}_I}\{\boldsymbol{\sigma}\}:\underline{[\![\mathbf{w}]\!]}+\int_{\mathcal{E}_D}\mathbf{w}\cdot\boldsymbol{\sigma}\boldsymbol{v}.$$

Next, noting that $\int_\Omega\mathbf{S}(\mathbf{w}):\boldsymbol{\sigma}=\int_\Omega\mathbf{S}(\mathbf{w}):\Pi_{\boldsymbol{\Sigma}_h}\boldsymbol{\sigma}$, applying the definition of $\mathbf{S}$ (cf. (2.22)), recalling that $[\![\sigma]\!]=0$ on $\mathcal{E}_I$, and using that $\mathbf{w}\cdot\tau v=\tau:(\mathbf{w}\otimes v)$, we arrive to

$$\begin{aligned}[A_h(\mathbf{u},\xi),(\mathbf{w},\eta)]+[B_h(\mathbf{w},\eta),p]-[F_h,(\mathbf{w},\eta)]&=\int_{\mathcal{E}_I}\{(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma})\}:\underline{[\![\mathbf{w}]\!]}-\int_{\mathcal{E}_I}([\![(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma})]\!]\otimes\boldsymbol{\beta}):\underline{[\![\mathbf{w}]\!]}\\ &\quad+\int_{\mathcal{E}_D}(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma}):(\mathbf{w}\otimes\boldsymbol{v}).\end{aligned}$$

Applying Cauchy–Schwarz's inequality, Lemmas 4.2 and 4.1, we get, with a constant $\bar{C}$ depending on $\widehat{\alpha}$ and l,

$$\begin{aligned}\left|\int_{\mathcal{E}_I}\{(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma})\}:\underline{[\![\mathbf{w}]\!]}\right|^2&\leqslant\bar{C}\|\mathrm{h}^{1/2}\{(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma})\}\|^2_{[L^2(\mathcal{E}_I)]^{2\times2}}\|\alpha^{1/2}\underline{[\![\mathbf{w}]\!]}\|^2_{[L^2(\mathcal{E}_I)]^{2\times2}}\\ &\leqslant\bar{C}\sum_{T\in\mathcal{T}_h}h_T\|(\mathbf{I}-\Pi_{\boldsymbol{\Sigma}_h})(\boldsymbol{\sigma})\|^2_{[L^2(\partial T)]^{2\times2}}|||\mathbf{w}|||_h^2\\ &\leqslant\bar{C}\sum_{T\in\mathcal{T}_h}h_T^{2\min\{t,r+1\}}\|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times2}}|||\mathbf{w}|||_h^2.\end{aligned}$$

The other integrals in the consistency term are bounded similarly to the previous one. We omit further details. □

The following lemma is also needed to derive the a-priori error estimate for **u**.

Lemma 4.4. *There exists* $C_{\mathrm{upp}} > 0$, *independent of the meshsize, such that*

$$|||\mathbf{v}|||_h^2 \leqslant C_{\mathrm{upp}} \sum_{T\in\mathcal{T}_h} \left\{ |\mathbf{v}|^2_{[H^1(T)]^2} + h_T^{-1} \|\mathbf{v}\|^2_{[L^2(\partial T)]^2} \right\} \quad \forall \mathbf{v} \in \mathbf{V}(h).$$

Proof. It is similar to the proof of Lemma 5.3 in [7]. □

Hence, as a consequence of the Strang-type error estimates (3.14) and (3.15) (cf. Theorem 3.1), and Lemmas 4.1, 4.3, and 4.4, we obtain the following result.

Theorem 4.1. *Let* $(\mathbf{t}, \mathbf{u}, \sigma, p, \xi)$ *and* $(\mathbf{t}_h, \mathbf{u}_h, \sigma_h, p_h, \xi_h)$ *be the solutions of* (1.4) *and* (2.15), *respectively. Assume that* $\mathbf{u}|_T \in [H^{t+1}(T)]^2$, $\sigma|_T \in [H^t(T)]^{2\times 2}$, *and* $p|_T \in H^t(T)$, *for all* $T \in \mathcal{T}_h$, *with* $t > 1/2$. *Then there exists* $C_{\mathrm{err}} > 0$, *independent of the meshsize, but depending on* $\widehat{\alpha}$, $\widehat{\beta}$, l, *and the polynomial approximation degree* k, *such that*

$$|||\mathbf{u}-\mathbf{u}_h|||_h^2 + \|p-p_h\|^2_{L^2(\Omega)} \leqslant C_{\mathrm{err}} \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k\}} \left\{ \|\mathbf{u}\|^2_{[H^{t+1}(T)]^2} + \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} + \|p\|^2_{H^t(T)} \right\}.$$

Proof. See Chapter 5 in [5] for details. □

The a-priori error estimate for the remaining unknowns **t** and $\boldsymbol{\sigma}$ is established next.

Theorem 4.2. *Assume the same hypotheses of Theorem* 4.1. *Then there exists* $\tilde{C}_{\mathrm{err}} > 0$, *independent of the meshsize, but depending on* $\widehat{\alpha}$, $\widehat{\beta}$, l, $C_{\mathbf{S}}$, *and the polynomial approximation degree* k, *such that*

$$\|\mathbf{t}-\mathbf{t}_h\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^{2\times 2}} \leqslant \tilde{C}_{\mathrm{err}} \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k\}} \left\{ \|\mathbf{u}\|^2_{[H^{t+1}(T)]^2} + \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} + \|p\|^2_{H^t(T)} \right\}.$$

Proof. It suffices to recall that $\mathbf{t} = \nabla\mathbf{u}, \mathbf{t}_h = \nabla_h\mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathbf{S}(\mathbf{u}), \boldsymbol{\sigma}_h = \Pi_{\Sigma_h}(\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I})$ and that $\sigma = \psi(\mathbf{t}) - p\mathbf{I}$, and then apply the a-priori error estimates for **u** and p provided by Theorem 4.1. We omit details and refer again to Chapter 5 in [5]. □

4.2. L^2-norm error estimate

We now turn our attention to the L^2-norm for the error $(\mathbf{u} - \mathbf{u}_h)$. To this end, we first recall from (3.6) that the Gâteaux derivative of $\mathcal{N}_h$ at any $(\mathbf{z},\zeta) \in \mathbf{V}(h)\times\mathbb{R}$ becomes

$$D\mathcal{N}_h(\mathbf{z},\zeta)((\mathbf{w},\eta),(\mathbf{v},\lambda)) := D\mathcal{N}(\boldsymbol{\varphi}(\mathbf{z},\zeta)+\mathcal{G})(\boldsymbol{\varphi}(\mathbf{w},\eta),\boldsymbol{\varphi}(\mathbf{v},\lambda)) \tag{4.5}$$

for all $(\mathbf{w},\eta),(\mathbf{v},\lambda) \in \mathbf{V}(h)\times\mathbb{R}$, with $\boldsymbol{\varphi}$ given by (3.4).

In what follows we assume that $\frac{\partial\psi_{ij}}{\partial\tilde{r}_{kl}}(\tilde{\mathbf{r}}) = \frac{\partial\psi_{kl}}{\partial\tilde{r}_{ij}}(\tilde{\mathbf{r}})$, for all $\tilde{\mathbf{r}} \in X$, and for all $i, j, k, l = 1, 2$, and that $D\mathcal{N}_h$ is *hemi-continuous*, that is for any $\mathbf{r}, \mathbf{s} \in X$, the mapping

$$\mathbb{R} \ni \mu \to D\mathcal{N}_h((\mathbf{w},\eta)+\mu(\mathbf{v},\lambda))((\mathbf{v},\lambda),\cdot) \in (\mathbf{V}(h)\times\mathbb{R})'$$

is continuous. Thus, applying the mean value theorem we deduce that there exists a convex combination of $(\mathbf{u},\xi)$ and $(\mathbf{u}_h,\xi_h)$, say $(\tilde{\mathbf{u}},\tilde{\xi}) \in \mathbf{V}(h)\times\mathbb{R}$, such that

$$D\mathcal{N}_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\mathbf{v},\lambda)) = [\mathcal{N}_h(\mathbf{u},\xi) - \mathcal{N}_h(\mathbf{u}_h,\xi_h),(\mathbf{v},\lambda)] \tag{4.6}$$

for all $(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbb{R}$. Further, it follows from (2.27), (3.7) and (4.6) that

$$DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\mathbf{v},\lambda)) = [A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\mathbf{v},\lambda)] \tag{4.7}$$

for all $(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbb{R}$.

Next, we let $(\mathbf{z},q)\in[H^1(\Omega)]^2\times L_0^2(\Omega)$ be the unique weak solution of the linear boundary value problem

$$\begin{aligned} &-\mathbf{div}\,\tilde{\boldsymbol{\sigma}} = \mathbf{u}-\mathbf{u}_h \quad \text{in } \Omega, \quad \operatorname{div}\mathbf{z}=0 \quad \text{in } \Omega, \quad \mathbf{z}=\mathbf{0} \quad \text{on } \Gamma,\\ &\tilde{\boldsymbol{\sigma}} := (\tilde{\sigma}_{ij}), \quad \tilde{\sigma}_{ij} := D\psi_{ij}(\boldsymbol{\varphi}(\tilde{\mathbf{u}},\tilde{\xi})+\mathscr{G}):\nabla\mathbf{z}-q\delta_{ij}, \end{aligned} \tag{4.8}$$

where $D\psi_{ij}(\tilde{\mathbf{r}})$ denotes the derivative (jacobian) of ψ_{ij} at $\tilde{\mathbf{r}}$, and assume that there exist $\gamma > 1/2$ and a constant $C_{\mathrm{reg}} > 0$, independent of $\mathbf{u}$ and $\mathbf{u}_h$, such that $\mathbf{z}\in[H^{\gamma+1}(\Omega)]^2\cap[H_0^1(\Omega)]^2, q\in H^\gamma(\Omega)\cap L_0^2(\Omega)$, and $\tilde{\boldsymbol{\sigma}}\in[H^\gamma(\Omega)]^{2\times 2}$, with

$$\|\tilde{\boldsymbol{\sigma}}\|_{[H^\gamma(\Omega)]^{2\times 2}} + \|\mathbf{z}\|_{[H^{\gamma+1}(\Omega)]^2} + \|q\|_{H^\gamma(\Omega)} \leqslant C_{\mathrm{reg}}\|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2}. \tag{4.9}$$

Hence, using the method applied in Section 2, we deduce that the mixed LDG formulation of problem (4.8) reduces to: *Find* $(\mathbf{z}_h,\zeta_h,q_h)\in\mathbf{V}_h\times\mathbb{R}\times W_h$ *such that*

$$\begin{aligned} &DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z}_h,\zeta_h),(\mathbf{v}_h,\lambda_h)) + [B_h(\mathbf{v}_h,\lambda_h),q_h] = [\tilde{F}_h,(\mathbf{v}_h,\lambda_h)],\\ &[B_h(\mathbf{z}_h,\zeta_h),r_h] = [\tilde{G}_h,r_h], \end{aligned} \tag{4.10}$$

for all $(\mathbf{v}_h,\lambda_h,r_h)\in\mathbf{V}_h\times\mathbb{R}\times W_h$, where B_h is given by (2.28), and the linear functionals $\tilde{F}_h:\mathbf{V}(h)\times\mathbb{R}\to\mathbb{R}, \tilde{G}_h:W_h\to\mathbb{R}$ are defined by

$$[\tilde{F}_h,(\mathbf{v},\lambda)] := \int_\Omega (\mathbf{u}-\mathbf{u}_h)\cdot\mathbf{v} \quad \forall(\mathbf{v},\lambda)\in\mathbf{V}(h)\times\mathbb{R} \tag{4.11}$$

and

$$[\tilde{G}_h,r_h] := 0 \quad \forall r_h\in W_h. \tag{4.12}$$

The unknown ζ_h corresponds to the discrete counterpart of the Lagrange multiplier ζ, which takes care of the uniqueness condition $\int_\Omega \operatorname{tr}\tilde{\boldsymbol{\sigma}}=0$. We remark that they are both zero.

As a consequence of the assumption (1.3) on ψ, and proceeding as in the proof of Lemma 3.1, one can show that $DA_h(\tilde{\mathbf{u}},\tilde{\xi})$ is uniformly $(\mathbf{V}(h)\times\mathbb{R})$-elliptic with respect to $\|\cdot\|_{\mathbf{V}(h)\times\mathbb{R}}$. In this way, since B_h satisfies the discrete inf-sup condition (cf. Lemma 3.3), we conclude that problem (4.10) has a unique solution $(\mathbf{z}_h,\zeta_h,q_h)\in\mathbf{V}_h\times\mathbb{R}\times W_h$. Furthermore, applying the linear version of the consistency estimate provided by Lemma 4.3, and using (4.9), we find that

$$\begin{aligned} |DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{w},\eta)) + [B_h(\mathbf{w},\eta),q] - [\tilde{F}_h,(\mathbf{w},\eta)]| &\leqslant C_{\mathrm{con}}h^{\min\{\gamma,r+1\}}\|\tilde{\boldsymbol{\sigma}}\|_{[H^\gamma(\Omega)]^{2\times 2}}|||\mathbf{w}|||_h\\ &\leqslant \tilde{C}_{\mathrm{con}}h^{\min\{\gamma,k\}}\|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2}|||\mathbf{w}|||_h \quad \forall(\mathbf{w},\eta)\in\mathbf{V}(h)\times\mathbb{R}, \end{aligned} \tag{4.13}$$

with $\tilde{C}_{\mathrm{con}} := C_{\mathrm{con}}C_{\mathrm{reg}}$, where the inequality $h^{\min\{\gamma,r+1\}}\leqslant h^{\min\{\gamma,k\}}$ has also been used.

The following theorem establishes the a-priori estimate for the L^2-norm of the error $(\mathbf{u}-\mathbf{u}_h)$.

Theorem 4.3. *Assume the hypotheses of Theorem* 4.1. *Then there exists* $\bar{C}_{\mathrm{err}}>0$, *independent of the meshsize, such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^2(\Omega)} \leqslant \bar{C}_{\mathrm{err}}h^{\min\{t,k\}+\min\{\gamma,k\}}\left\{\sum_{T\in\mathscr{T}_h}\left(\|\mathbf{u}\|^2_{[H^{t+1}(T)]^2}+\|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}}+\|p\|^2_{H^t(T)}\right)\right\}^{1/2}.$$

Proof. Taking $(\mathbf{v},\lambda) := (\mathbf{u}-\mathbf{u}_h, \xi-\xi_h) \in \mathbf{V}(h)\times\mathbb{R}$ in (4.11), and adding and substracting convenient expressions, we can write

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{[L^2(\Omega)]^2} &= [\tilde{F}_h, (\mathbf{u}-\mathbf{u}_h, \xi-\xi_h)] \\ &= DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) + [B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q] \\ &\quad - \Big(DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) + [B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q] - [\tilde{F}_h,(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)]\Big),\end{aligned}$$

which, according to (4.13), and using that $\xi = \xi_h = 0$, yields

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{[L^2(\Omega)]^2} &\leqslant |DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) + [B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q]| \\ &\quad + \tilde{C}_{\mathrm{con}} h^{\min\{\gamma,k\}} \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2} |||\mathbf{u}-\mathbf{u}_h|||_h. \end{aligned} \tag{4.14}$$

It is easy to check that $[B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),r_h] = 0$ for all $r_h \in W_h$. Hence, applying the boundedness of B_h (with constant C_{bh}), Lemma 4.1, and the estimate (4.9), we find that

$$\begin{aligned}|[B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),q]| &= |[B_h(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\mathbf{I}-\Pi_{W_h})(q)]| \\ &\leqslant C_{\mathrm{bh}} |||\mathbf{u}-\mathbf{u}_h|||_h \|(\mathbf{I}-\Pi_{W_h})(q)\|_{L^2(\Omega)} \\ &\leqslant \bar{C}_{\mathrm{con}} h^{\min\{\gamma,k\}} |||\mathbf{u}-\mathbf{u}_h|||_h \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2}, \end{aligned} \tag{4.15}$$

with $\bar{C}_{\mathrm{con}} = C_{\mathrm{bh}} C_{\mathrm{ort}} C_{\mathrm{reg}}$.

Next, employing the symmetry of $\mathcal{D}A_h(\tilde{\mathbf{u}},\tilde{\xi})$, adding and substracting $(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)$, and denoting $e_h(\mathbf{z}) = (\mathbf{I}-\Pi_{\mathbf{V}_h})(\mathbf{z})$, we obtain

$$\begin{aligned}DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) &= DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)) \\ &\quad + DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{u}-\mathbf{u}_h,\xi-\xi_h),(e_h(\mathbf{z}),\zeta-\zeta_h)),\end{aligned}$$

which, thanks to (4.7), becomes

$$\begin{aligned}DA_h(\tilde{\mathbf{u}},\tilde{\xi})((\mathbf{z},\zeta),(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h)) &= [A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] \\ &\quad + [A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(e_h(\mathbf{z}),\zeta-\zeta_h)]. \end{aligned} \tag{4.16}$$

Now, applying Lemma 4.4, (4.3) and the approximation properties provided in Lemma 4.1, and using the regularity estimate (4.9), we get

$$|||e_h(\mathbf{z})|||_h^2 \leqslant 2C_{\mathrm{upp}}C_{\mathrm{ort}}^2 h^{2\min\{\gamma,k\}} \|\mathbf{z}\|^2_{[H^{\gamma+1}(\Omega)]^2} \leqslant 2C_{\mathrm{upp}}C_{\mathrm{ort}}^2 C_{\mathrm{reg}}^2 h^{2\min\{\gamma,k\}} \|\mathbf{u}-\mathbf{u}_h\|^2_{[L^2(\Omega)]^2}. \tag{4.17}$$

Thus, the Lipschitz-continuity of A_h and (4.17) imply that

$$\begin{aligned}[A_h(\mathbf{u},\xi)-A_h(\mathbf{u}_h,\xi_h),(e_h(\mathbf{z}),\zeta-\zeta_h)] &\leqslant C_{\mathrm{LC}} |||\mathbf{u}-\mathbf{u}_h|||_h |||e_h(\mathbf{z})|||_h \\ &\leqslant \hat{C}_{\mathrm{con}} h^{\min\{\gamma,k\}} |||\mathbf{u}-\mathbf{u}_h|||_h \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2}, \end{aligned} \tag{4.18}$$

with a positive constant $\hat{C}_{\mathrm{con}}$ depending on C_{LC}, C_{upp}, C_{ort}, and C_{reg}.

On the other hand, since $\operatorname{div}\mathbf{z} = 0$, $\mathbf{S}(\mathbf{z}) = \mathbf{0}$, and $\zeta = 0$, we find that

$$[B_h(\mathbf{z},\zeta),r] = 0 \quad \forall r \in L^2(\Omega). \tag{4.19}$$

In addition, using that $\operatorname{tr}(\nabla\mathbf{z}) = \operatorname{div}\mathbf{z} = 0$, and applying Gauss' formula, noting that $\mathbf{z}\in[H^{\gamma+1}(\Omega)]^2$, $\mathbf{z} = \mathbf{0}$ on Γ, and $\mathbf{div}(\psi(\nabla\mathbf{u}) - p\mathbf{I}) = -\mathbf{f}$, we obtain

$$[A_h(\mathbf{u},\xi),(\mathbf{z},\zeta)] = \int_\Omega \boldsymbol{\psi}(\nabla\mathbf{u}) : \nabla\mathbf{z} = \int_\Omega (\boldsymbol{\psi}(\nabla\mathbf{u}) - p\mathbf{I}) : \nabla\mathbf{z} = \int_\Omega \mathbf{f}\cdot\mathbf{z} = [F_h,(\mathbf{z},\zeta)]. \tag{4.20}$$

Also, according to the first equation of the mixed formulation (2.26), we have that

$$[A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] + [B_h(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h),p_h] = [F_h,(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)]. \tag{4.21}$$

In this way, replacing $[A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)]$ by the expression derived from (4.21), and inserting $0 = [F_h,(\mathbf{z},\zeta)] - [A_h(\mathbf{u},\xi),(\mathbf{z},\zeta)]$ from (4.20), we can write

$$\begin{aligned}[A_h(\mathbf{u},\xi) - A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] &= [A_h(\mathbf{u},\xi),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] - [F_h,(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] + [B_h(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h),p_h] \\ &\quad + [F_h,(\mathbf{z},\zeta)] - [A_h(\mathbf{u},\xi),(\mathbf{z},\zeta)],\end{aligned}$$

which, employing from (4.19) that $[B_h(\mathbf{z},\zeta),p_h] = 0$, yields

$$\begin{aligned}[A_h(\mathbf{u},\xi) - A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] &= [B_h(e_h(\mathbf{z}),\zeta-\zeta_h),p-p_h] \\ &\quad - \{[A_h(\mathbf{u},\xi),(e_h(\mathbf{z}),\zeta-\zeta_h)] + [B_h(e_h(\mathbf{z}),\zeta-\zeta_h),p] - [F_h,(e_h(\mathbf{z}),\zeta-\zeta_h)]\}.\end{aligned}$$

The boundedness of B_h, the consistency estimate given by Lemma 4.3, and the fact that $h^{\min\{t,r+1\}} \leqslant h^{\min\{t,k\}}$, imply that

$$[A_h(\mathbf{u},\xi) - A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] \leqslant C_{\mathrm{bh}}|||e_h(\mathbf{z})|||_h \|p-p_h\|_{L^2(\Omega)} + C_{\mathrm{con}} h^{\min\{t,k\}} \left\{ \sum_{T\in\mathcal{T}_h} \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} \right\}^{1/2} |||e_h(\mathbf{z})|||_h,$$

which, thanks to (4.17), becomes

$$\begin{aligned}[A_h(\mathbf{u},\xi) - A_h(\mathbf{u}_h,\xi_h),(\Pi_{\mathbf{V}_h}(\mathbf{z}),\zeta_h)] &\leqslant \tilde{C} h^{\min\{\gamma,k\}} \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2} \|p-p_h\|_{L^2(\Omega)} \\ &\quad + \tilde{C} h^{\min\{t,k\}+\min\{\gamma,k\}} \left\{ \sum_{T\in\mathcal{T}_h} \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} \right\}^{1/2} \|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2},\end{aligned} \tag{4.22}$$

with a positive constant $\tilde{C}$ depending on C_{bh}, C_{con}, C_{upp}, C_{ort}, and C_{reg}.

Finally, (4.14)–(4.16), (4.18), and (4.22) give

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{[L^2(\Omega)]^2} &\leqslant \left(\tilde{C}_{\mathrm{con}} + \bar{C}_{\mathrm{con}} + \hat{C}_{\mathrm{con}} + \tilde{C}\right) h^{\min\{\gamma,k\}} \left\{ |||\mathbf{u}-\mathbf{u}_h|||_h + \|p-p_h\|_{L^2(\Omega)} \right\} \\ &\quad + \tilde{C} h^{\min\{t,k\}+\min\{\gamma,k\}} \left\{ \sum_{T\in\mathcal{T}_h} \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} \right\}^{1/2},\end{aligned}$$

which, together with the estimates for $|||\mathbf{u}-\mathbf{u}_h|||_h$ and $\|p-p_h\|_{L^2(\Omega)}$ provided in Theorem 4.1, completes the proof. □

5. A-posteriori error analysis

Hereafter, we consider problem (1.1) with homogeneous Dirichlet condition, that is $\mathbf{g} = \mathbf{0}$. Then we redefine $\mathbf{V}(h)$ as $\mathbf{V}(h) := \mathbf{V}_h + [H_0^1(\Omega)]^2$ and introduce the semilinear global operator $\mathbf{A}_h : (\mathbf{V}(h)\times\mathbb{R}\times L^2(\Omega)) \to (\mathbf{V}(h)\times\mathbb{R}\times L^2(\Omega))'$ and the linear functional $\mathbf{F}_h \in (\mathbf{V}(h)\times\mathbb{R}\times L^2(\Omega))'$ arising after adding the two equations in (2.26), that is

$$[\mathbf{A}_h(\mathbf{w},\eta,r),(\mathbf{v},\lambda,q)] := [A_h(\mathbf{w},\eta),(\mathbf{v},\lambda)] + [B_h(\mathbf{v},\lambda),r] + [B_h(\mathbf{w},\eta),q] \tag{5.1}$$

and

$$[\mathbf{F}_h, (\mathbf{v}, \lambda, q)] := [F_h, (\mathbf{v}, \lambda)] + [G_h, q] \tag{5.2}$$

for all $(\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q) \in \mathbf{V}(h) \times \mathbb{R} \times L^2(\Omega)$.

It follows easily that the Gâteaux derivative of $\mathbf{A}_h$ at $(\mathbf{z}, \zeta, s) \in \mathbf{V}(h) \times \mathbb{R} \times L^2(\Omega)$ reduces to the bounded bilinear form $D\mathbf{A}_h(\mathbf{z}, \zeta, s) : (\mathbf{V}(h) \times \mathbb{R} \times L^2(\Omega)) \times (\mathbf{V}(h) \times \mathbb{R} \times L^2(\Omega)) \to \mathbb{R}$ defined by

$$D\mathbf{A}_h(\mathbf{z}, \zeta, s)((\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q)) := DA_h(\mathbf{z}, \zeta)((\mathbf{w}, \eta), (\mathbf{v}, \lambda)) + [B_h(\mathbf{v}, \lambda), r] + [B_h(\mathbf{w}, \eta), q] \tag{5.3}$$

for all $(\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q) \in \mathbf{V}(h) \times \mathbb{R} \times L^2(\Omega)$, where (cf. (3.7) and (3.4))

$$DA_h(\mathbf{z}, \zeta)((\mathbf{w}, \eta), (\mathbf{v}, \lambda)) := D\mathcal{N}(\boldsymbol{\varphi}(\mathbf{z}, \zeta) + \mathcal{G})(\boldsymbol{\varphi}(\mathbf{w}, \eta), \boldsymbol{\varphi}(\mathbf{v}, \lambda)) + \int_{\mathcal{E}_I} \alpha \underline{[\![\mathbf{w}]\!]} : \underline{[\![\mathbf{v}]\!]} + \int_{\mathcal{E}_D} \alpha(\mathbf{w} \otimes \mathbf{v}) : (\mathbf{v} \otimes \mathbf{v}) \tag{5.4}$$

and $\boldsymbol{\varphi}(\mathbf{v},\lambda) := \nabla_h \mathbf{v} - \mathbf{S}(\mathbf{v}) - \lambda \mathbf{I}$ for all $(\mathbf{v}, \lambda) \in \mathbf{V}(h) \times \mathbb{R}$.

The derivation of our a-posteriori error estimate in Theorem 5.1 below will make use of an inf-sup condition for $D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)$ and a consistency error estimate (in terms of $\mathbf{A}_h$ and $\mathbf{F}_h$) for problem (2.26). More precisely, the following two lemmas are needed.

Lemma 5.1. *Let $(\tilde{\mathbf{u}}, \tilde{\xi}) \in \mathbf{V}(h) \times \mathbb{R}$ be such that (4.6) and (4.7) hold. Then there exist $C, \tilde{C} > 0$, independent of the meshsize and $(\tilde{\mathbf{u}}, \tilde{\xi})$, such that for any $(\mathbf{w}, \eta, r) \in [H_0^1(\Omega)]^2 \times \mathbb{R} \times L_0^2(\Omega)$ there exists $(\mathbf{v}, \lambda, q) \in [H_0^1(\Omega)]^2 \times \mathbb{R} \times L_0^2(\Omega)$ satisfying*

$$D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q)) \geqslant C\|(\mathbf{w}, \eta, r)\|_{\mathrm{LDG}}^2 \tag{5.5}$$

and

$$\|(\mathbf{v}, \lambda, q)\|_{\mathrm{LDG}} \leqslant \tilde{C}\|(\mathbf{w}, \eta, r)\|_{\mathrm{LDG}}. \tag{5.6}$$

Proof. We adapt the proof of Lemma 4.3 in [19] to the present situation. In fact, given $(\mathbf{w}, \eta, r) \in [H_0^1(\Omega)]^2 \times \mathbb{R} \times L_0^2(\Omega)$ we first observe, according to Corollary 2.4 in [17], that there exists $\mathbf{z} \in [H_0^1(\Omega)]^2$ such that

$$-\int_\Omega r \operatorname{div} \mathbf{z} \geqslant C_0\|r\|_{L^2(\Omega)}^2 \quad \text{and} \quad \||\mathbf{z}\||_h \leqslant \|r\|_{L^2(\Omega)}. \tag{5.7}$$

Then, we choose $\mathbf{v} := \kappa_0\mathbf{w} + \kappa_1\mathbf{z}$, $q := -\kappa_0 r$, and $\lambda := \kappa_0\eta$, where κ_0 and κ_1 are positive constants to be determined so that (5.5) and (5.6) hold. Since $\mathbf{w}, \mathbf{v} \in [H_0^1(\Omega)]^2$, we have that $\mathbf{S(w)} = \mathbf{S(v)} = \mathbf{0}$, $\underline{[\![\mathbf{w}]\!]} = \underline{[\![\mathbf{v}]\!]} = \mathbf{0}$ on $\mathcal{E}_I$, and $\mathbf{v} = \mathbf{0}$ on $\mathcal{E}_D$. It follows from (5.3), (5.4), and the definition of B_h (cf. (2.28)), that

$$\begin{aligned} D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q)) = {} & \kappa_1 D\mathcal{N}(\boldsymbol{\varphi}(\tilde{\mathbf{u}}, \tilde{\xi}) + \mathcal{G})(\nabla\mathbf{w} - \eta\mathbf{I}, \nabla\mathbf{z}) \\ & + \kappa_0 D\mathcal{N}(\boldsymbol{\varphi}(\tilde{\mathbf{u}}, \tilde{\xi}) + \mathcal{G})(\nabla\mathbf{w} - \eta\mathbf{I}, \nabla\mathbf{w} - \eta\mathbf{I}) - \kappa_1 \int_\Omega r \operatorname{div} \mathbf{z}, \end{aligned}$$

which, applying (3.3), (5.7), and the inequality $-ab \geqslant -\frac{a^2}{2\epsilon} - \frac{\epsilon b^2}{2}$, yields

$$D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q)) \geqslant c_1(\epsilon)\|\nabla\mathbf{w} - \eta\mathbf{I}\|_{[L^2(\Omega)]^{2\times 2}}^2 + c_2(\epsilon)\|r\|_{L^2(\Omega)}^2 \quad \forall \epsilon > 0,$$

where $c_1(\epsilon) := \kappa_0\tilde{C}_2 - \frac{\kappa_1\tilde{C}_1}{2\epsilon}$ and $c_2(\epsilon) := \kappa_1 C_0 - \frac{\kappa_1\epsilon\tilde{C}_1}{2}$.

Hence, observing from (3.9) that $\|\nabla\mathbf{w} - \eta\mathbf{I}\|_{[L^2(\Omega)]^{2\times 2}}^2 = \|\nabla\mathbf{w}\|_{[L^2(\Omega)]^{2\times 2}}^2 + 2|\Omega||\eta|^2$, and taking $\epsilon = \frac{C_0}{\tilde{C}_1}$, $\kappa_0 = 1$, and $\kappa_1 = \frac{\tilde{C}_2 C_0}{\tilde{C}_1^2}$, the above inequality becomes

$$D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{w}, \eta, r), (\mathbf{v}, \lambda, q)) \geqslant \frac{\tilde{C}_2}{2}\|\nabla\mathbf{w}\|_{[L^2(\Omega)]^{2\times 2}}^2 + \tilde{C}_2|\Omega||\eta|^2 + \frac{\tilde{C}_2 C_0^2}{2\tilde{C}_1^2}\|r\|_{L^2(\Omega)}^2, \tag{5.8}$$

which proves (5.5). Finally, the estimate (5.6) is a direct consequence of the choice of $(\mathbf{v}, \lambda, q)$ and the upper bound for $|||\mathbf{z}|||_h$ in (5.7). □

Lemma 5.2. *Let* $(\mathbf{v},q) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ *and define the orthogonal projections* $\mathbf{v}_h := \Pi_{\mathbf{V}_h}(\mathbf{v}) \in \mathbf{V}_h$ *and* $q_h := \Pi_{W_h}(q) \in W_h \cap L_0^2(\Omega)$. *Then, there exists a constant* $C_{\text{con}} > 0$, *independent of h, such that*

$$|[\mathbf{F}_h, (\mathbf{v}-\mathbf{v}_h, 0, q-q_h)] - [\mathbf{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v}-\mathbf{v}_h, 0, q-q_h)]| \leqslant C_{\text{con}} \boldsymbol{\eta} \|(\mathbf{v}, 0, q)\|_{\text{LDG}}, \tag{5.9}$$

where $\boldsymbol{\eta}^2 := \sum_{T\in\mathcal{T}_h} \eta_T^2$, *and for each* $T \in \mathcal{T}_h$

$$\begin{aligned}\eta_T^2 := {} & h_T^2 \|\mathbf{f} + \mathbf{div}\, \boldsymbol{\psi}(\mathbf{t}_h) - \nabla p_h\|_{[L^2(T)]^2}^2 + \|\mathrm{tr}(\mathbf{t}_h)\|_{[L^2(T)]^2}^2 + h_T \|[\![\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}]\!]\|_{[L^2(\partial T/\Gamma)]^2}^2 \\ & + h_T \|\boldsymbol{\sigma}_h - (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I})\|_{[L^2(\partial T\cap\mathcal{E}_D)]^{2\times 2}}^2 + \|\alpha^{1/2}\mathbf{u}_h \otimes \boldsymbol{\nu}\|_{[L^2(\partial T\cap\mathcal{E}_D)]^{2\times 2}}^2 \\ & + h_T \|\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta} - \{\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}\}\|_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}^2 + \|\alpha^{1/2}\underline{[\![\mathbf{u}_h]\!]}\|_{[L^2(\partial T\cap\mathcal{E}_I)]^{2\times 2}}^2.\end{aligned} \tag{5.10}$$

Proof. We first note, according to the definitions of A_h and B_h (cf. (2.27), (2.28)), that

$$\begin{aligned}[A_h(\mathbf{u}_h, \xi_h), (\mathbf{v}-\mathbf{v}_h, 0)] + [B_h(\mathbf{v}-\mathbf{v}_h, 0), p_h] = {} & \int_\Omega (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) : (\nabla_h(\mathbf{v}-\mathbf{v}_h) - \mathbf{S}(\mathbf{v}-\mathbf{v}_h)) \\ & + \int_{\mathcal{E}_I} \alpha \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} + \int_{\mathcal{E}_D} \alpha(\mathbf{u}_h \otimes \boldsymbol{\nu}) : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}),\end{aligned}$$

which, applying integration by parts and using that $\boldsymbol{\sigma}_h = \Pi_{\Sigma_h}(\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I})$ (cf. (2.25)), yields

$$\begin{aligned}[A_h(\mathbf{u}_h, \xi_h), (\mathbf{v}-\mathbf{v}_h, 0)] + [B_h(\mathbf{v}-\mathbf{v}_h, 0), p_h] = {} & \sum_{T\in\mathcal{T}_h} \Bigg(-\int_T \mathrm{div}(\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) \cdot (\mathbf{v}-\mathbf{v}_h) \\ & + \int_{\partial T} (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}) \Bigg) + \int_{\mathcal{E}_I} \alpha \underline{[\![\mathbf{u}_h]\!]} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} \\ & + \int_{\mathcal{E}_D} \alpha(\mathbf{u}_h \otimes \boldsymbol{\nu}) : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}) - \int_\Omega \boldsymbol{\sigma}_h : \mathbf{S}(\mathbf{v}-\mathbf{v}_h).\end{aligned} \tag{5.11}$$

Next, simple computations show that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \int_{\partial T} (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}) = {} & \int_{\mathcal{E}_I} \{\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}\} : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} + \int_{\mathcal{E}_I} [\![\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}]\!] \cdot \{\mathbf{v}-\mathbf{v}_h\} \\ & + \int_{\mathcal{E}_D} (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}).\end{aligned} \tag{5.12}$$

In addition, the definition of $\mathbf{S}$ (cf. (2.22)) and the fact that $(\mathbf{v}-\mathbf{v}_h) \cdot \sigma_h \nu = \sigma_h : (\mathbf{v}-\mathbf{v}_h) \otimes \nu$, imply that

$$\int_\Omega \boldsymbol{\sigma}_h : \mathbf{S}(\mathbf{v}-\mathbf{v}_h) = \int_{\mathcal{E}_I} (\{\boldsymbol{\sigma}_h\} - [\![\boldsymbol{\sigma}_h]\!] \otimes \boldsymbol{\beta}) : \underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} + \int_{\mathcal{E}_D} \boldsymbol{\sigma}_h : ((\mathbf{v}-\mathbf{v}_h) \otimes \boldsymbol{\nu}). \tag{5.13}$$

Hence, replacing (5.12) and (5.13) back into (5.11), we find that

$$
\begin{aligned}
[A_h(\mathbf{u}_h,\xi_h),(\mathbf{v}-\mathbf{v}_h,0)]+[B_h(\mathbf{v}-\mathbf{v}_h,0),p_h] = & -\int_\Omega \mathbf{div}_h(\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I})\cdot(\mathbf{v}-\mathbf{v}_h)+\int_{\mathscr{E}_I}\alpha\underline{[\![\mathbf{u}_h]\!]}:\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} \\
& +\int_{\mathscr{E}_D}\alpha(\mathbf{u}_h\otimes\boldsymbol{\nu}):((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}) \\
& -\int_{\mathscr{E}_I}(\{\boldsymbol{\sigma}_h\}-[\![\boldsymbol{\sigma}_h]\!]\otimes\boldsymbol{\beta}-\{\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}\}):\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} \\
& -\int_{\mathscr{E}_D}(\boldsymbol{\sigma}_h-\boldsymbol{\psi}(\mathbf{t}_h)+p_h\mathbf{I}):((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}) \\
& +\int_{\mathscr{E}_I}[\![\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}]\!]\cdot\{\mathbf{v}-\mathbf{v}_h\}. \qquad (5.14)
\end{aligned}
$$

On the other hand, using that $\mathbf{t}_h=\nabla_h\mathbf{u}_h-\mathbf{S}(\mathbf{u}_h)+\mathscr{G}$ (cf. (2.24)) and that $\mathbf{g}=0$ in the present case, we obtain

$$
[B_h(\mathbf{u}_h,\xi_h),q-q_h]=-\int_\Omega(q-q_h)\mathbf{I}:(\mathbf{t}_h-\mathscr{G})=-\int_\Omega(q-q_h)\mathrm{tr}(\mathbf{t}_h). \qquad (5.15)
$$

It follows from (5.14) and (5.15) that

$$
\begin{aligned}
&[\mathbf{F}_h,(\mathbf{v}-\mathbf{v}_h,0,q-q_h)]-[\mathbf{A}_h(\mathbf{u}_h,\xi_h,p_h),(\mathbf{v}-\mathbf{v}_h,0,q-q_h)] \\
&\quad=\sum_{T\in\mathscr{T}_h}\int_T(\mathbf{f}+\mathbf{div}\,\boldsymbol{\psi}(\mathbf{t}_h)-\nabla p_h)\cdot(\mathbf{v}-\mathbf{v}_h)+\sum_{T\in\mathscr{T}_h}\int_T(q-q_h)\mathrm{tr}(\mathbf{t}_h)-\int_{\mathscr{E}_I}[\![\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}]\!]\cdot\{\mathbf{v}-\mathbf{v}_h\} \\
&\quad-\int_{\mathscr{E}_I}\alpha\underline{[\![\mathbf{u}_h]\!]}:\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}+\int_{\mathscr{E}_I}(\{\boldsymbol{\sigma}_h\}-[\![\boldsymbol{\sigma}_h]\!]\otimes\boldsymbol{\beta}-\{\boldsymbol{\psi}(\mathbf{t}_h)-p_h\mathbf{I}\}):\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]} \\
&\quad+\int_{\mathscr{E}_D}(\boldsymbol{\sigma}_h-\boldsymbol{\psi}(\mathbf{t}_h)+p_h\mathbf{I}):((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu})+\int_{\mathscr{E}_D}\alpha(\mathbf{u}_h\otimes\boldsymbol{\nu}):((\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}),
\end{aligned}
$$

which, applying the Cauchy–Schwarz inequality, implies that

$$
|[\mathbf{F}_h,(\mathbf{v}-\mathbf{v}_h,\lambda-\lambda_h,q-q_h)]-[\mathbf{A}_h(\mathbf{u}_h,\xi_h,p_h),(\mathbf{v}-\mathbf{v}_h,\lambda-\lambda_h,q-q_h)]|\leqslant C\boldsymbol{\eta}H(\mathbf{v},q)^{1/2},
$$

with

$$
\begin{aligned}
H(\mathbf{v},q):=&\sum_{T\in\mathscr{T}_h}h_T^{-2}\|\mathbf{v}-\mathbf{v}_h\|^2_{[L^2(T)]^2}+\sum_{T\in\mathscr{T}_h}\|q-q_h\|^2_{L^2(T)}+\|\alpha^{1/2}\{\mathbf{v}-\mathbf{v}_h\}\|^2_{[L^2(\mathscr{E}_I)]^2} \\
&+\|\alpha^{1/2}\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}\|^2_{[L^2(\mathscr{E}_I)]^{2\times2}}+\|\alpha^{1/2}(\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}\|^2_{[L^2(\mathscr{E}_D)]^{2\times2}}.
\end{aligned}
$$

Now, applying Lemma 4.2 (see also Lemma 3.1 in [7]) and the approximation property (4.2) (cf. Lemma 4.1), we obtain

$$
\begin{aligned}
&\|\alpha^{1/2}\{\mathbf{v}-\mathbf{v}_h\}\|^2_{[L^2(\mathscr{E}_I)]^2}+\|\alpha^{1/2}\underline{[\![\mathbf{v}-\mathbf{v}_h]\!]}\|^2_{[L^2(\mathscr{E}_I)]^{2\times2}}+\|\alpha^{1/2}(\mathbf{v}-\mathbf{v}_h)\otimes\boldsymbol{\nu}\|^2_{[L^2(\mathscr{E}_D)]^{2\times2}} \\
&\quad\leqslant C\sum_{T\in\mathscr{T}_h}h_T\|\mathrm{h}^{-1}(\mathbf{v}-\mathbf{v}_h)\|^2_{[L^2(\partial T)]^2}\leqslant C\sum_{T\in\mathscr{T}_h}\|\mathbf{v}\|^2_{[H^1(T)]^2}=C\|\mathbf{v}\|^2_{[H^1(\Omega)]^2}.
\end{aligned}
$$

Similarly, applying the approximation property (4.1) (cf. Lemma 4.1) and observing that $\|q-q_h\|_{L^2(\Omega)}=\|q-\Pi_{W_h}(q)\|_{L^2(\Omega)}\leqslant\|q\|_{L^2(\Omega)}$, we get

$$
\sum_{T\in\mathscr{T}_h}h_T^{-2}\|\mathbf{v}-\mathbf{v}_h\|^2_{[L^2(T)]^2}+\sum_{T\in\mathscr{T}_h}\|q-q_h\|^2_{L^2(T)}\leqslant C\|v\|^2_{[H^1(\Omega)]^2}+\|q\|_{L^2(\Omega)}.
$$

The last two inequalities and the fact that $\mathbf{v} \in [H_0^1(\Omega)]^2$ show that $H(\mathbf{v},q)$ is bounded above by $\widetilde{C}_{\text{con}}\|(\mathbf{v},0,q)\|_{\text{LDG}}$, where $\widetilde{C}_{\text{con}}$ is a positive constant independent of the meshsize. This provides (5.9) and finishes the proof. □

We are now in a position to establish the main result of this section.

Theorem 5.1. *There exists a constant* $C_{\text{rel}} > 0$, *independent of the meshsize*, *such that*

$$\|(\mathbf{t}-\mathbf{t}_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\mathbf{u}-\mathbf{u}_h,\xi-\xi_h,p-p_h)\| \leqslant C_{\text{rel}}\boldsymbol{\vartheta}, \tag{5.16}$$

where $\boldsymbol{\vartheta}^2 := \sum_{T\in\mathscr{T}_h}\vartheta_T^2$ *and the local error estimator* ϑ_T *is given by*

$$\vartheta_T^2 := \eta_T^2 + |T||\bar{p}_h|^2 + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|^2_{[L^2(T)]^{2\times 2}}, \tag{5.17}$$

with $\bar{p}_h$ *being the mean value of* p_h.

Proof. Since $\mathbf{t} = \nabla\mathbf{u}$ and $\mathbf{t}_h = \nabla_h\mathbf{u}_h - \mathbf{S}(\mathbf{u}_h) + \mathbf{S}(\mathbf{u})$, we easily obtain, applying (2.23), that

$$\|\mathbf{t}-\mathbf{t}_h\|^2_{[L^2(\Omega)]^{2\times 2}} \leqslant 2\max\{1, C_{\mathbf{S}}^2\}|||\mathbf{u}-\mathbf{u}_h|||_h^2.$$

Also, replacing $\boldsymbol{\sigma}$ by $\psi(\mathbf{t}) - p\mathbf{I}$, adding and substracting $\psi(\mathbf{t}_h) - p_h\mathbf{I}$, and then applying triangle inequality and the Lipschitz-continuity of the nonlinear operator induced by ψ, we obtain

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^{2\times 2}} &= \|\boldsymbol{\psi}(\mathbf{t}) - p\mathbf{I} - (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) + (\boldsymbol{\psi}(\mathbf{t}_h) - p_h\mathbf{I}) - \boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^{2\times 2}}\\ &\leqslant C\Big\{\|\mathbf{t}-\mathbf{t}_h\|_{[L^2(\Omega)]^{2\times 2}} + \|p-p_h\|_{L^2(\Omega)} + \|\boldsymbol{\sigma}_h - \boldsymbol{\psi}(\mathbf{t}_h) + p_h\mathbf{I}\|_{[L^2(\Omega)]^{2\times 2}}\Big\}.\end{aligned}$$

It follows from the above inequalities that it remains to estimate $\|(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h,p-p_h)\|^2_{\text{LDG}}$. To this end, we first let $\mathbf{V}_0^\perp$ be the orthogonal complement of $\mathbf{V}_0 := \mathbf{V}_h \cap [H_0^1(\Omega)]^2$ within $\mathbf{V}_h$ with respect to the inner product inducing the norm $|||\cdot|||_h$, and recall from [20] that $|\cdot|_h$ and $|||\cdot|||_h$ are equivalent on $\mathbf{V}_0^\perp$ with constants independent of h. Hence, in what follows we write $\mathbf{u}_h = \mathbf{u}_h^0 + \mathbf{u}_h^\perp$, with $\mathbf{u}_h^0 \in \mathbf{V}_0$ and $\mathbf{u}_h^\perp \in \mathbf{V}_0^\perp$. Also, we write $p_h = p_{h,0} + \bar{p}_h$, with $p_{h,0} \in L_0^2(\Omega)$ and $\bar{p}_h \in \mathbb{R}$.

A simple application of triangle inequality and the above mentioned equivalence between $|\cdot|_h$ and $|||\cdot|||_h$, yields

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h,p-p_h)\|^2_{\text{LDG}} &\leqslant 2\Big\{\|(\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0})\|^2_{\text{LDG}} + |||\mathbf{u}_h^\perp|||_h^2 + |\Omega||\bar{p}_h|^2\Big\}\\ &\leqslant c\Big\{\|(\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0})\|^2_{\text{LDG}} + |\mathbf{u}_h^\perp|_h^2 + |\Omega||\bar{p}_h|^2\Big\},\end{aligned}$$

which, using that $\underline{[\![\mathbf{u}_h^\perp]\!]} = \underline{[\![\mathbf{u}_h]\!]}$ on $\mathscr{E}_I$ and $\mathbf{u}_h^\perp = \mathbf{u}_h$ on $\mathscr{E}_D$, becomes

$$\|(\mathbf{u}-\mathbf{u}_h,\xi-\xi_h,p-p_h)\|^2_{\text{LDG}} \leqslant c\Big\{\|(\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0})\|^2_{\text{LDG}} + |\mathbf{u}_h|_h^2 + |\Omega||\bar{p}_h|^2\Big\}. \tag{5.18}$$

Now, since $p \in L_0^2(\Omega)$ we apply Lemma 5.1 to $(\mathbf{w},\eta,r) := (\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0}) \in [H_0^1(\Omega)]^2 \times \mathbb{R}\times L_0^2(\Omega)$, and deduce that there exists $(\mathbf{v},\lambda,q) \in [H_0^1(\Omega)]^2\times\mathbb{R}\times L_0^2(\Omega)$ such that

$$C\|(\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0})\|^2_{\text{LDG}} \leqslant D\mathbf{A}_h(\tilde{\mathbf{u}},\tilde{\xi},p)((\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0}),(\mathbf{v},\lambda,q)) \tag{5.19}$$

and

$$\|(\mathbf{v},\lambda,q)\|_{\text{LDG}} \leqslant \widetilde{C}\|(\mathbf{u}-\mathbf{u}_h^0,\xi-\xi_h,p-p_{h,0})\|_{\text{LDG}}, \tag{5.20}$$

with C and $\widetilde{C} > 0$ independent of the meshsize.

Then, setting $\mathbf{v}_h = \Pi_{\mathbf{V}_h}(\mathbf{v}) \in \mathbf{V}_h, \lambda_h = \lambda \in \mathbb{R}$, and $q_h = \Pi_{W_h}(q) \in W_h \cap L_0^2(\Omega)$, we easily obtain

$$\begin{aligned} D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u} - \mathbf{u}_h^0, \xi - \xi_h, p - p_{h,0}), (\mathbf{v}, \lambda, q)) &= D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v}, \lambda, q)) \\ &\quad + D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u}_h^\perp, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)) \\ &= D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)) \\ &\quad + D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u} - \mathbf{u}_h, \xi - \xi_h, p - p_h), (\mathbf{v}_h, \lambda_h, q_h)) \\ &\quad + D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u}_h^\perp, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)), \end{aligned}$$

which, applying (5.3), (4.7), and the definition of $\mathbf{A}_h$ (cf. (5.1)), becomes

$$\begin{aligned} &= [\mathbf{A}_h(\mathbf{u}, \xi, p), (\mathbf{v}, \lambda, q)] - [\mathbf{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v}_h, \lambda_h, q_h)] - [\mathbf{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)] \\ &\quad + D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u}_h^\perp, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)). \end{aligned}$$

Since $(\mathbf{u}_h, \xi_h, p_h)$ is the solution of (2.26) and $(\mathbf{v}, \lambda, q) \in [H_0^1(\Omega)]^2 \times \mathbb{R} \times L_0^2(\Omega)$, we find, respectively, that

$$[\mathbf{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v}_h, \lambda_h, q_h)] = [\mathbf{F}_h, (\mathbf{v}_h, \lambda_h, q_h)]$$

and

$$[\mathbf{A}_h(\mathbf{u}, \xi, p), (\mathbf{v}, \lambda, q)] = [\mathbf{F}_h, (\mathbf{v}, \lambda, q)],$$

whence

$$\begin{aligned} D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u} - \mathbf{u}_h^0, \xi - \xi_h, p - p_{h,0}), (\mathbf{v}, \lambda, q)) &= D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)((\mathbf{u}_h^\perp, 0, \bar{p}_h), (\mathbf{v}, \lambda, q)) + [\mathbf{F}_h, (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)] \\ &\quad - [\mathbf{A}_h(\mathbf{u}_h, \xi_h, p_h), (\mathbf{v} - \mathbf{v}_h, 0, q - q_h)]. \end{aligned}$$

Finally, the above expression is bounded above by applying Lemma 5.2, the uniform boundedness of $D\mathbf{A}_h(\tilde{\mathbf{u}}, \tilde{\xi}, p)$, the equivalence between $|\cdot|_h$ and $|||\cdot|||_h$ in $\mathbf{V}_0^\perp$, and the estimate (5.20). The resulting terms are replaced back into (5.19) and (5.18), thus completing the proof. We omit further details. □

6. Numerical results

In this section, we provide several numerical results illustrating the performance of the mixed LDG method and the fully explicit a-posteriori error estimate $\boldsymbol{\vartheta}$. We emphasize that the actual computations are carried on the original discrete system (2.15) and not on the equivalent reduced one (2.26), which, as explained before, was introduced just for theoretical reasons.

Hereafter, N is the number of degrees of freedom defining the subspace $\mathbf{\Sigma}_h \times \mathbf{V}_h \times \mathbf{\Sigma}_h \times W_h \times \mathbb{R}$, that is $N := C_\kappa \times (\text{number of triangles of } \mathcal{T}_h) + 1$, with $C_\kappa = 15{,}39$ for the $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximations, respectively. In addition, the individual and global errors are defined as follows

$$\mathbf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{[L^2(\Omega)]^{2\times 2}}, \quad \mathbf{e}_h(\mathbf{u}) := |||\,\mathbf{u} - \mathbf{u}_h|||_h, \quad \mathbf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\Omega)]^{2\times 2}},$$

$$\mathbf{e}(p) := \|p - p_h\|_{L^2(\Omega)} \quad \text{and} \quad \mathbf{e} := \left\{[\mathbf{e}(\mathbf{t})]^2 + [\mathbf{e}_h(\mathbf{u})]^2 + [\mathbf{e}(\boldsymbol{\sigma})]^2 + [\mathbf{e}(p)]^2\right\}^{1/2},$$

where $(\mathbf{t}_h, \mathbf{u}_h, \boldsymbol{\sigma}_h, p_h, \xi_h) \in \boldsymbol{\Sigma}_h \times \mathbf{V}_h \times \boldsymbol{\Sigma}_h \times W_h \times \mathbb{R}$ is the unique solution of the discrete scheme (2.15). Also, if $\mathbf{e}$ and $\tilde{\mathbf{e}}$ stand for the error at two consecutive triangulations with N and $\tilde{N}$ degrees of freedom, respectively, then we define the associated experimental rate of convergence by

$$r := -2\frac{\log(\mathbf{e}/\tilde{\mathbf{e}})}{\log(N/\tilde{N})}. \tag{6.1}$$

On the other hand, the adaptive algorithm used in the mesh refinement process, without *hanging nodes*, is the following [26]:

1. Start with a coarse mesh $\mathscr{T}_h$.
2. Solve the discrete problem (2.15) for the actual mesh $\mathscr{T}_h$.
3. Compute ϑ_T for each triangle $T \in \mathscr{T}_h$.
4. Evaluate stopping criterion and decide to finish or go to next step.
5. Use *red–blue–green* procedure to refine each $T' \in \mathscr{T}_h$ whose error estimator $\vartheta_{T'}$ satisfies $\vartheta_{T'} \geqslant \frac{1}{2}\max\{\vartheta_T : T \in \mathscr{T}_h\}$.
6. Define resulting mesh as actual mesh $\mathscr{T}_h$ and go to step 2.

The numerical results presented below were obtained in a *Compaq Alpha ES40 Parallel Computer* using a MATLAB code. We remark that in the pure nonlinear case, the corresponding mixed LDG scheme (cf. (2.15)), which becomes a nonlinear algebraic system with N unknowns, is solved by Newton–Raphson's method with the initial guess given by the solution of the associated linear Stokes problem, and setting the tolerance in 10^{-3} for the relative error. In all cases we take the parameters $\widehat{\alpha} = 1$ and $\boldsymbol{\beta} = (1,1)^{t}$ in

Table 6.1
Example 1 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation: uniform, red–blue–green, and red refinements

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	ϑ	$\mathbf{e}/\vartheta$	r
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	–
361	81.5164	18.2308	30.1984	17.0232	300.2485	0.3012	–
5761	61.3113	20.9879	24.0423	8.2928	151.1932	0.4604	0.5341
23041	34.9934	12.5703	13.9487	4.2752	82.1369	0.4863	0.8016
92161	19.2402	6.7162	7.4175	2.2261	44.6317	0.4885	0.8736
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	–
271	81.5063	18.2332	30.0414	16.8825	300.2306	0.3009	–
631	61.4417	20.9911	24.0254	8.2642	151.2558	0.4610	2.2073
991	23.5086	8.0024	9.0209	2.9443	53.1132	0.5005	4.2206
5011	7.7918	2.5289	2.8891	0.9878	17.5121	0.4992	1.2401
13636	4.7580	1.5850	1.8217	0.6350	10.9793	0.4894	0.9724
22321	3.6281	1.2182	1.3953	0.4810	8.4037	0.4881	1.0957
52306	2.3582	0.8008	0.9246	0.3269	5.5424	0.4829	1.0026
85216	1.8248	0.6250	0.7164	0.2476	4.3067	0.4812	1.0482
91	27.7530	4.5119	8.6094	5.1848	104.5580	0.2856	–
181	81.2676	18.2260	30.0517	16.8955	300.3732	0.3001	–
361	61.2733	20.9934	23.9869	8.2052	151.3981	0.4594	2.5811
451	36.1299	13.1283	14.5617	4.4548	86.7567	0.4766	4.6732
1441	14.3907	5.1411	5.6798	1.7073	34.7893	0.4712	1.2382
6841	6.3317	2.1388	2.3950	0.7621	14.8890	0.4796	1.0490
14941	4.2436	1.4197	1.5912	0.5081	9.9563	0.4797	1.0294
30826	2.9866	0.9939	1.1213	0.3672	7.0034	0.4800	0.9701
64801	2.0519	0.6803	0.7640	0.2458	4.8001	0.4804	1.0146

Table 6.2
Example 1 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation: uniform, red–blue–green, and red refinements

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	ϑ	$\mathbf{e}/\vartheta$	r
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	–
937	76.7277	14.0911	23.8389	13.5966	301.5960	0.2742	–
3745	79.0125	14.8175	20.6097	10.1293	237.0900	0.3526	–
14977	36.4816	7.7792	9.2608	3.5529	94.4255	0.4088	1.1152
59905	10.7937	2.5907	2.8672	0.8686	27.1349	0.4237	1.7473
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	–
703	76.6581	14.0920	23.8200	13.5795	301.6019	0.2740	–
1639	36.5359	7.7802	9.2614	3.5526	94.4537	0.4092	4.5989
2107	11.0779	2.6417	2.9362	0.9063	27.9778	0.4216	9.4498
8035	1.4901	0.2517	0.2829	0.0913	2.6754	0.5757	1.5892
16732	0.6428	0.1099	0.1241	0.0407	1.1777	0.5648	2.2897
41692	0.2781	0.0435	0.0498	0.0173	0.4703	0.6090	1.7394
62323	0.1755	0.0273	0.0306	0.0098	0.2922	0.6178	2.2958
235	25.8203	8.3819	13.2488	7.2551	106.6109	0.2914	–
469	76.6090	14.1048	23.8151	13.5686	301.6354	0.2738	–
937	36.5155	7.7958	9.2642	3.5390	94.5687	0.4085	5.3744
2107	6.2777	1.5374	1.6880	0.4929	15.8892	0.4216	2.4275
4681	2.3065	0.5519	0.6094	0.1827	5.6086	0.4378	1.8943
7840	1.2740	0.2883	0.3206	0.0991	2.8517	0.4729	2.3237
12988	0.7608	0.1751	0.1922	0.0560	1.7232	0.4677	2.0399
20125	0.5138	0.1131	0.1240	0.0359	1.1099	0.4881	1.8143
32878	0.3149	0.0678	0.0732	0.0195	0.6576	0.5032	2.0084
51247	0.2050	0.0434	0.0468	0.0123	0.4205	0.5116	1.9407

the corresponding dual formulation. In addition, we test our results considering both regular meshes and meshes with *hanging nodes*, though the latter is not covered yet by the theory. In this case, our refinement strategy is similar to the one described before, but instead of using the *red–blue–green* procedure in step 5, we apply the *red* one.

We present two examples. In the first one, we consider the linear version of the boundary value problem (1.1), that is the usual Stokes model, in the L-shaped domain $\Omega := (-1,1)^2 \backslash [0,1]^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

$$\begin{cases} \mathbf{u}(\mathbf{x}) := \left(-\sqrt{1000}\,\mathrm{e}^{-\sqrt{1000}(x_1+x_2)}, \sqrt{1000}\,\mathrm{e}^{-\sqrt{1000}(x_1+x_2)}\right), \\ p(\mathbf{x}) := 2\mathrm{e}^{x_1}\sin(x_2) - \frac{2}{3}(\mathrm{e}-1)(\cos(1)-1) \end{cases}$$

for all $\mathbf{x} := (x_1,x_2)^{\mathrm{t}} \in \Omega$. We notice that $\mathbf{u}$ is divergence free in Ω and presents an inner layer around the origin.

The second example deals with the pure nonlinear case, where the kinematic viscosity function ψ is given by the Carreau law, that is $\psi(t) = \kappa_0 + \kappa_1(1 + t^2)^{(\beta-2)/2}$. It is easy to check that ψ satifies (1.2) and (1.3) for all $\kappa_0,\kappa_1 > 0$, and for all $\beta \in [1,2]$. Note that the usual linear Stokes model is obtained with $\beta = 2$. In this example, we take $\kappa_0 = \kappa_1 = 1/2$ and $\beta = 3/2$, whence $\psi(t) := \frac{1}{2} + \frac{1}{2}(1+t^2)^{-1/4}$. In addition, we consider again the L-shaped domain $\Omega := (-1,1)^2 \backslash [0,1]^2$, and choose $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given by

Table 6.3
Example 2 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation: uniform, red–blue–green, and red refinements

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	ϑ	$\mathbf{e}/\vartheta$	r
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	–
361	7.7723	2.2596	2.3950	1.2706	18.4319	0.4631	0.2323
1441	6.6210	2.1460	1.8748	0.8721	13.8864	0.5229	0.2338
5761	5.5502	1.9470	1.4139	0.5316	10.6885	0.5681	0.2579
23041	4.0926	1.6475	1.0679	0.3190	7.2802	0.6250	0.4164
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	0.4030
151	7.6738	2.2881	2.5494	1.3567	19.1942	0.4435	0.6431
1441	6.0212	2.0920	1.6023	0.6301	12.1643	0.5428	0.3432
2521	4.0010	1.6850	1.4065	0.6498	7.8272	0.5889	1.8819
5476	1.9541	0.8959	0.8945	0.4484	4.0282	0.5886	1.1239
14311	1.2336	0.5882	0.5500	0.2618	2.5071	0.5968	0.9586
30286	0.8324	0.4209	0.3819	0.1720	1.7441	0.5863	1.0157
58306	0.6252	0.3041	0.2717	0.1218	1.2784	0.5916	0.9207
91	9.2999	2.0266	2.6848	1.5977	24.0305	0.4169	–
136	7.8666	2.3018	2.5023	1.3009	19.5785	0.4427	0.7203
1036	6.2889	2.2420	1.8157	0.7571	12.7759	0.5448	0.3240
1576	4.5476	1.9310	1.6943	0.7919	8.7614	0.6030	1.8744
1801	3.7327	1.6846	1.6247	0.8088	7.6311	0.5870	2.4720
2566	3.0437	1.4010	1.3076	0.6200	5.8273	0.6263	1.1571
9271	1.7748	0.7662	0.6885	0.3130	3.1381	0.6615	0.8856
16741	1.3377	0.5519	0.4940	0.2204	2.3413	0.6598	0.9998
31411	0.9979	0.4150	0.3614	0.1546	1.7145	0.6708	0.9379
57376	0.7468	0.3000	0.2623	0.1120	1.2706	0.6720	0.9885

$$\begin{cases} \mathbf{u}(\mathbf{x}) := \left[(x_1 - 0.01)^2 + (x_2 - 0.01)^2\right]^{-1/2}(x_2 - 0.01, 0.01 - x_1), \\ p(\mathbf{x}) := \frac{1}{1.1 - x_1} - \frac{1}{3}\ln\left(\frac{441}{11}\right) \end{cases}$$

for all $\mathbf{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$. We observe here that $\mathbf{u}$ is divergence free in Ω and singular in an exterior neighborhood of (0,0). In addition, p is singular in an exterior neighborhood of the segment $\{1\} \times [0,1]$.

In Tables 6.1–6.4, we summarize the individual errors, the error estimate ϑ, the effectivity index $\mathbf{e}/\vartheta$, and the corresponding experimental rates of convergence for the uniform and adaptive refinements associated to Examples 1 and 2 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximations. The errors on each triangle were computed applying a 7 points Gaussian quadrature rule. We notice that the effectivity indexes are bounded above and below, which confirm the reliability of ϑ, and provide numerical evidences for their efficiency, even in the case of irregular meshes. In addition, Figs. 6.1–6.4 display the global errors $\mathbf{e}$, $\mathbf{e}^{\mathrm{rbg}}$, and $\mathbf{e}^{\mathrm{r}}$, corresponding to the uniform, red–blue–green, and red refinements, respectively, versus the degrees of freedom N. In all cases the errors of the adaptive methods decrease much faster than those of the uniform ones, which is emphasized by the experimental rates of convergence provided in Tables 6.1–6.4, showing that the adaptive algorithms recover $O(h)$ and $O(h^2)$ for $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ and $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$, respectively. Equivalently, as observed from the definition

Table 6.4
Example 2 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation: uniform, red–blue–green, and red refinements

N	$\mathbf{e}_h(\mathbf{u})$	$\mathbf{e}(\mathbf{t})$	$\mathbf{e}(\boldsymbol{\sigma})$	$\mathbf{e}(p)$	ϑ	$\mathbf{e}/\vartheta$	r
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	–
937	8.0976	1.2593	1.4422	0.8134	13.2850	0.6293	0.4929
3745	5.8418	1.1927	1.0402	0.5114	9.8167	0.6187	0.4612
14977	4.5820	1.1203	0.7805	0.3071	7.6189	0.6288	0.3423
59905	3.1715	0.9091	0.5735	0.1874	4.7975	0.6991	0.5145
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	–
391	7.7162	1.3267	1.9996	1.2351	14.2108	0.5752	1.4273
2965	5.4469	1.2465	1.0177	0.4753	9.5063	0.5996	0.3364
4369	4.5954	1.0963	0.8193	0.3499	7.5857	0.6338	0.8780
6826	1.9690	0.6159	0.4188	0.1580	2.6850	0.7863	4.9650
8464	1.1428	0.2966	0.2538	0.1205	1.6572	0.7323	5.1478
12910	0.4847	0.1622	0.1342	0.0591	0.7527	0.7064	3.9094
35608	0.1745	0.0572	0.0473	0.0207	0.2692	0.7086	2.3166
55420	0.1286	0.0394	0.0333	0.0142	0.1871	0.7446	1.4201
235	11.2569	1.3855	2.5978	1.6797	19.4592	0.6041	–
1288	6.5150	1.3518	1.3646	0.7214	11.3569	0.6014	0.3359
3160	3.5570	1.0168	0.8412	0.3877	6.5021	0.5865	3.7560
5617	1.3036	0.3821	0.3174	0.1407	1.9539	0.7176	3.9263
7957	0.8613	0.3030	0.2504	0.1080	1.3152	0.7245	2.2179
14041	0.5171	0.1569	0.1295	0.0566	0.7749	0.7208	1.8812
23635	0.2992	0.1059	0.0834	0.0344	0.4531	0.7282	2.0220
33346	0.2023	0.0655	0.0531	0.0220	0.3055	0.7211	2.3469
49843	0.1463	0.0462	0.0370	0.0152	0.2082	0.7615	1.6377

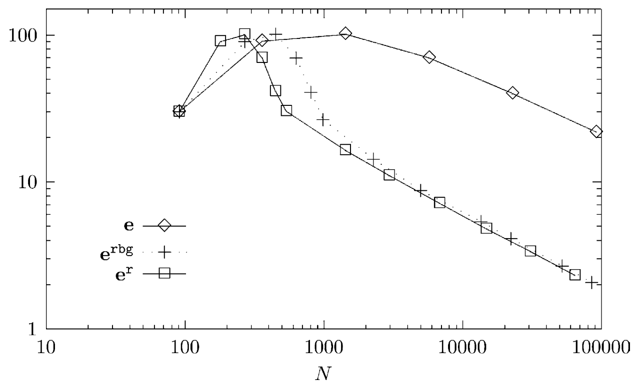

Fig. 6.1. Example 1 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation: global error $\mathbf{e}$ for the uniform and adaptive refinements.

of r (cf. (6.1)), the slope of the curves displayed in Figs. 6.1–6.4, measured every two consecutive points, is given by $-r/2$.

Next, Figs. 6.5–6.8 display some intermediate meshes obtained with the different refinements. As expected, the adaptive algorithms are able to recognize the inner layer of Example 1 and the singularities of $\mathbf{u}$ and p in Example 2. In addition, we notice that the good behaviour of the red refinement (with

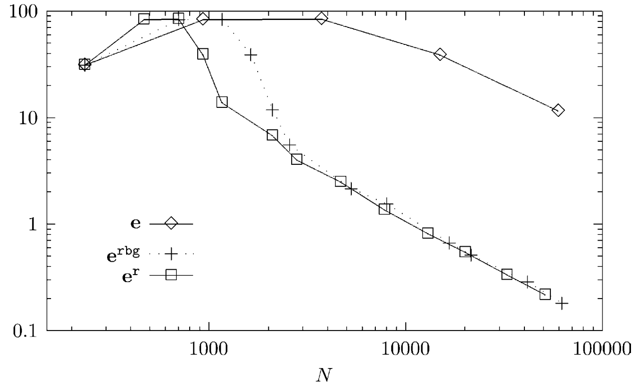

Fig. 6.2. Example 1 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation: global error **e** for the uniform and adaptive refinements.

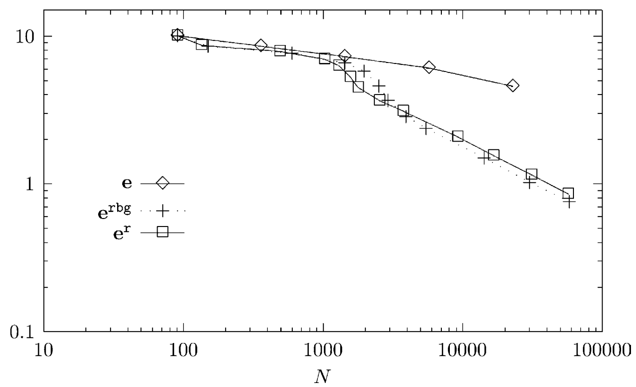

Fig. 6.3. Example 2 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation: global error **e** for the uniform and adaptive refinements.

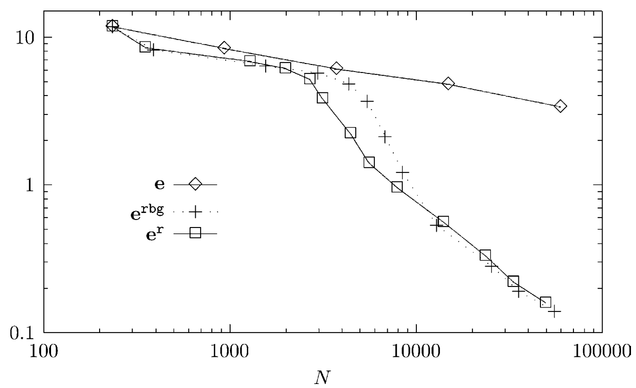

Fig. 6.4. Example 2 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation: global error **e** for the uniform and adaptive refinements.

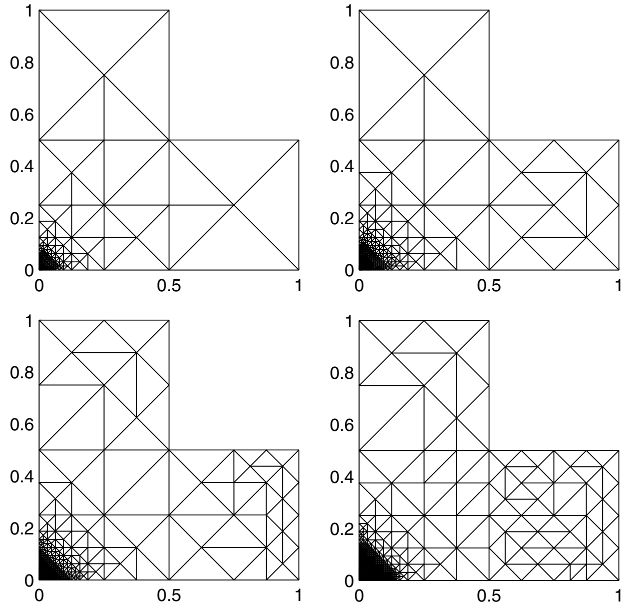

Fig. 6.5. Example 1 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation, without hanging nodes: adapted intermediate meshes with 5011, 13 636, 22 321 and 52 306 degrees of freedom.

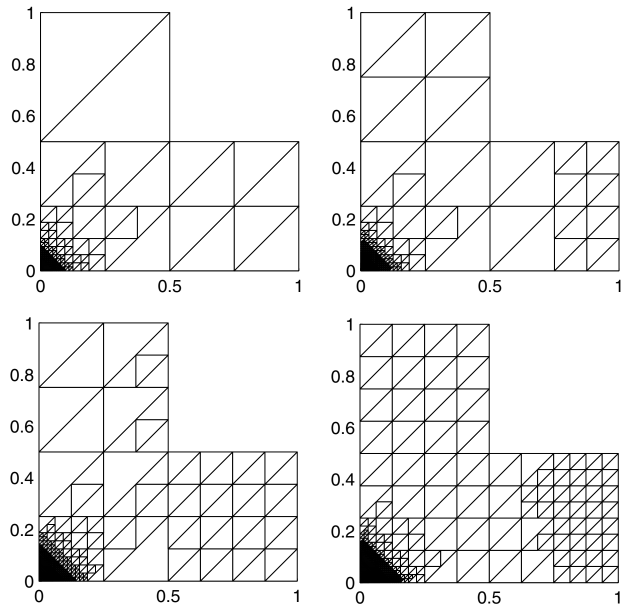

Fig. 6.6. Example 1 with $\mathbb{P}_0 - \mathbb{P}_0 - \mathbb{P}_1 - \mathbb{P}_0$ approximation, with hanging nodes: adapted intermediate meshes with 6841, 14 941, 30 826 and 64 801 degrees of freedom.

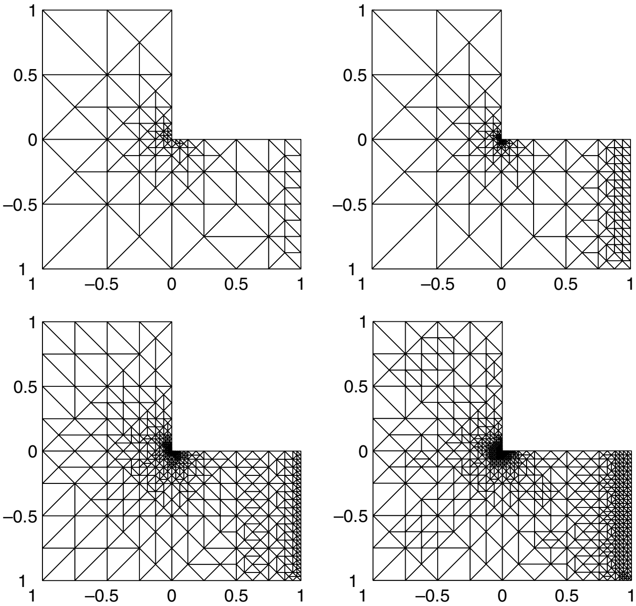

Fig. 6.7. Example 2 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation, without hanging nodes: adapted intermediate meshes with 6826, 12910, 35608 and 55420 degrees of freedom.

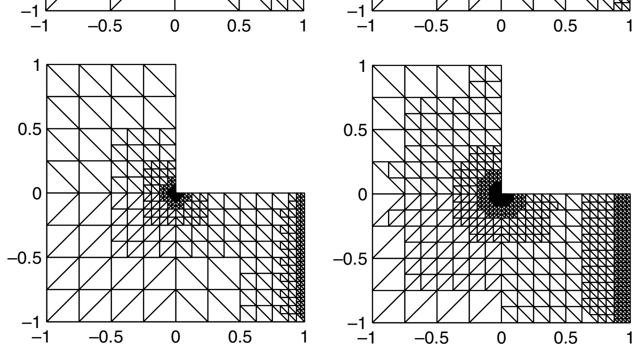

Fig. 6.8. Example 2 with $\mathbb{P}_1 - \mathbb{P}_1 - \mathbb{P}_2 - \mathbb{P}_1$ approximation, with hanging nodes: adapted intermediate meshes with 5617, 14041, 23635 and 49843 degrees of freedom.

hanging nodes) give us numerical evidences that our results are still valid on this kind of meshes. In particular, we observe that the red refinement is more localized around the singularities than the red–blue–green one.

References

[1] D.N. Arnold, F. Brezzi, B. Cockburn, L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM Journal on Numerical Analysis 39 (5) (2001) 1749–1779.

[2] R. Becker, P. Hansbo, M.G. Larson, Energy norm a posteriori error estimation for discontinuous Galerkin methods, Computer Methods in Applied Mechanics and Engineering 192 (2003) 723–733.

[3] R. Becker, P. Hansbo, R. Stenberg, A finite element method for domain decomposition with non-matching grids, Preprint 2001-15, Chalmers Finite Element Center, Chalmers University of Technology, Sweden, 2001.

[4] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, Berlin Heidelberg New York, 1991.

[5] R. Bustinza, Numerical analysis of transmission problems with discontinuities (spanish), Ph.D. thesis, Universidad de Concepción, Concepción, Chile, 2004.

[6] R. Bustinza, B. Cockburn, G.N. Gatica, An a-posteriori error estimate for the local discontinuous Galerkin method applied to linear and nonlinear diffusion problems, Journal of Scientific Computing 22 (1), in press.

[7] R. Bustinza, G.N. Gatica, A local discontinuous Galerkin method for nonlinear diffusion problems with mixed boundary conditions, SIAM Journal on Scientific Computing 26 (1) (2004) 152–177.

[8] P. Castillo, B. Cockburn, I. Perugia, D. Schötzau, An a priori error analysis of the local discontinuous Galerkin method for elliptic problems, SIAM Journal on Numerical Analysis 38 (5) (2000) 1676–1706.

[9] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[10] B. Cockburn, G. Kanschat, D. Schötzau, The local discontinuous Galerkin method for linear incompressible fluid flow: a review, Preprint 03-10, Department of Mathematics, University of Basel, 2003.

[11] B. Cockburn, G. Kanschat, D. Schötzau, The local discontinuous Galerkin method for the Oseen equations, Mathematics of Computation 73 (2004) 569–593.

[12] B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab, Local discontinuous Galerkin methods for the Stokes system, SIAM Journal on Numerical Analysis 40 (1) (2002) 319–343.

[13] G.N. Gatica, M. González, S. Medahhi, A low-order mixed finite element method for a class of quasi-Newtonian Stokes flows. Part I: a-priori error analysis, Computer Methods in Applied Mechanics and Engineering 193 (9–11) (2004) 881–892.

[14] G.N. Gatica, M. González, S. Medahhi, A low-order mixed finite element method for a class of quasi-Newtonian Stokes flows. Part II: a-posteriori error analysis, Computer Methods in Applied Mechanics and Engineering 193 (9–11) (2004) 893–911.

[15] G.N. Gatica, N. Heuer, S. Meddahi, On the numerical analysis of nonlinear two-fold saddle point problems, IMA Journal of Numerical Analysis 23 (2) (2003) 301–330.

[16] G.N. Gatica, F.J. Sayas, A note on the local approximation properties of piecewise polynomials with applications to LDG methods, Preprint 2004-06, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, 2004.

[17] V. Girault, P.A. Raviart, Finite element methods for Navier–Stokes equations: theory and algorithms, Springer Series in Computational Mathematics 5 (1986).

[18] P. Houston, J. Robson, E. Süli, Discontinuous Galerkin finite element approximation of quasilinear elliptic boundary value problems I: the scalar case, Preprint NA-04/14, Numerical Analysis Group, Computing Laboratory, Oxford University, UK, 2004.

[19] P. Houston, D. Schötzau, T.P. Wihler, Energy norm a posteriori error estimation for mixed discontinuous Galerkin approximations of the Stokes problem, Journal of Scientific Computing 22(1), in press.

[20] O.A. Karakashian, F. Pascal, A-posteriori error estimates for a discontinuous Galerkin approximation of second order elliptic problems, SIAM Journal on Numerical Analysis 41 (6) (2003) 2374–2399.

[21] H. Manouzi, M. Farhloul, Mixed finite element analysis of a non-linear three-fields Stokes model, IMA Journal of Numerical Analysis 21 (2001) 143–164.

[22] I. Perugia, D. Schötzau, The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations, Mathematics of Computation 72 (2003) 1179–1214.

[23] B. Riviere, M.F. Wheeler, A posteriori error estimates and mesh adaptation strategy for discontinuous Galerkin methods applied to diffusion problems, Preprint 00-10, TICAM, University of Texas at Austin, USA, 2000.

[24] J.E. Roberts, J.-M. Thomas, Mixed and hybrid methods, in: P.G. Ciarlet, J.L. Lions (Eds.), Handbook of Numerical Analysis, Finite Element Methods (Part 1), vol. II, North-Holland, Amsterdam, 1991.

[25] D. Schötzau, C. Schwab, A. Toselli, Mixed hp-DGFEM for incompressible flows, SIAM Journal on Numerical Analysis 40 (6) (2003) 2171–2194.

[26] R. Verfürth, A Review of a Posteriori Error Estimation and Adaptive Mesh-refinement Techniques, Wiley-Teubner, Chichester, 1996.